Complexité avancée - TD 3

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Exercise 1 (Restrictions of the SAT problem). *1. Let 3-SAT be the restriction of SAT to clauses consisting of at most three literals (called 3-clauses). In other words, the input is a finite set S of 3-clauses, and the question is whether S is satisfiable. Show that 3-SAT is NP-complete for logspace reductions (assuming SAT is).*

2. *Let 2-SAT be the restriction of SAT to clauses consisting of at most two literals (called 2-clauses). Show that 2-SAT is in P, using proofs by resolution.*

3. *Show that 2-UNSAT (i.e, the unsatisfiability of a set of 2-clauses) is NL-complete.*

4. *Conclude that 2-SAT is NL-complete. You may use the fact that* $\mathsf{co}-\mathsf{NL} = \mathsf{NL}$.

We recall the space-hierarchy theorem.

Theorem 1 (Space-hierarchy theorem). *For two space-constructible functions f and g such that $f = o(g)$, we have* $\mathsf{DSPACE}(f) \subsetneq \mathsf{DSPACE}(g)$.

Exercise 2 (Poly-logarithmic space). *1. Let* $\mathsf{polyL} = \cup_{k\in\mathbb{N}}\mathsf{SPACE}(\log^k)$. *Show that polyL does not have a complete problem for logarithmic space reduction.*[1]

2. *Recall that* $\mathsf{PSPACE} = \cup_{k\in\mathbb{N}}\mathsf{SPACE}(n^k)$. *Does PSPACE have a complete problem for logarithmic space reduction ? Why doesn't the proof of the previous question apply to PSPACE?*

Exercise 3 (Padding argument). *1. Show that if* $\mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$ *for some $c > 0$, then* $\mathsf{PSPACE} \subseteq \mathsf{NP}$.

Hint: for Ł $\in \mathsf{DSPACE}(n^k)$ one may consider the language $\tilde{L} = \{(x, w_x) \mid x \in L\}$. where w_x is a word written in unary.

2. *Deduce that* $\mathsf{DSPACE}(n^c) \neq \mathsf{NP}$.

Exercise 4 (On the existence of One-way function). *A one-way function is a bijection f from k-bit integers to k-bit integers such that f is computable in polynomial time, but f^{-1} is not. Prove that for all one-way functions f, we have*

$$A := \{(x, y) \mid f^{-1}(x) < y\} \in (\mathsf{NP} \cap \mathsf{coNP})\backslash\mathsf{P}$$

Exercise 5 (Regular languages). *Let REG denote the set regular/rational languages.*

1. *Show that for all $L \in \mathsf{REG}$, L is recognized by a TM running in space 0 and time $n + 1$.*[2]

[1] Note that, from this, we can deduce that $\mathsf{polyL} \neq \mathsf{P}$.

[2] In fact, regular languages exactly correspond to languages that can be recognized in such a way.

2. *Exhibit a language recognized by a TM running in space* $\log n$ *and time* $O(n)$ *that is not in* REG.

Exercise 6 (Yet another NL-complete problem)**.** *For a finite set* X*, a subset* $S \subseteq X$*, and a binary operator* $* : X \times X \to X$ *defined on* X*, we inductively define* $S_{0,*} := S$ *and* $S_{i+1,*} := S_{i,*} \cup \{x * y \mid x, y \in S_{i,*}\}$*. The closure of* S *with regard to* $*$ *is the set* $S_* = \cup_{i \in \mathbb{N}} S_{i,*}$.

Show that the following problem is NL*-complete.*

- *Input: A finite set* X*, a binary operation* $* : X \times X \to X$ *that is associative (i.e.* $(x * y) * z = x * (y * z)$ *for all* $x, y, z \in X$*), a subset* $S \subseteq X$ *and a target* $t \in X$*.*
- *Output: Yes if and only* $t \in S_*$*.*