Game Theory

Week 2 — Strategic games, mixed strategies, and minimax

Conventions. Players are indexed by superscripts: player i has strategy set S^i and utility u^i. In bimatrix games, player 1 chooses a row and player 2 a column. When a game is zero-sum, only the payoff of the row player is shown. Mixed strategies are identified with probability vectors.

Exercise 1 — Bot arena: dominance and equilibrium.

Two autonomous agents play a one-shot game. Player 1 chooses a row and player 2 a column; their utilities are given by

	X	Y	Z
A	$(4,4)$	$(4,3)$	$(0,1)$
B	$(3,1)$	$(3,3)$	$(2,2)$
C	$(1,0)$	$(1,5)$	$(1,4)$

(a) Determine the best responses of each player to every pure strategy of the other player. Find all pure Nash equilibria.

(b) Which strategies are strictly dominated in the original game? Eliminate any such strategies and continue eliminating strictly dominated strategies whenever possible.

(c) There is more than one possible order for the first eliminations. Check that every maximal sequence of eliminations of strictly dominated strategies leads to the same reduced game.

(d) Prove the following general fact: a strictly dominated strategy can never be used in a Nash equilibrium. Deduce that iterated elimination of strictly dominated strategies cannot eliminate a Nash equilibrium.

Exercise 2 — Weak dominance.

Consider the game

	c_1	c_2
r_1	$(2,6)$	$(3,2)$
r_2	$(0,2)$	$(3,2)$

(a) Mark the best responses of both players and determine all pure Nash equilibria. Which of them are strict equilibria?

(b) Show that c_2 is weakly dominated. Eliminate it and find the Nash equilibria of the reduced game.

(c) What happened to the Nash equilibria of the original game? Explain precisely why the argument from Exercise 1(d) breaks down for weak dominance.

(d) Prove the following robustness property: a strict Nash equilibrium cannot be eliminated by removing weakly dominated strategies. More precisely, show by induction that it survives every sequence of eliminations of weakly dominated strategies.

Exercise 3 — Mixed strategies.

Consider a two-player zero-sum game whose payoff matrix for the row player is

$$A = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}.$$

Let p be the probability with which the row player chooses the first row.

(a) Write the row player's expected payoff against each of the three pure columns as a function of p. Draw the three functions for $0 \le p \le 1$.

(b) For every p, determine the payoff that the row player can guarantee. Hence draw the lower envelope

$$L(p) = \min_j u(p, c_j)$$

and find a mixed strategy that maximises it.

(c) Let now A be an arbitrary $m \times n$ payoff matrix and fix a mixed row strategy $x \in \Delta_m$. Prove

$$\min_{y \in \Delta_n} x^\top A y = \min_{1 \le j \le n} x^\top A e_j.$$

In words: against a fixed mixed strategy, a worst response can always be chosen pure. State and prove the analogous assertion for a best response to a fixed mixed strategy.

(d) Explain geometrically why, when the row player has two pure strategies, the function

$$p \longmapsto \min_j u(p, c_j)$$

must attain its maximum. How does the picture generalise when the row player has three pure strategies? If you know the relevant topological terminology, formulate the general argument for an arbitrary finite game.

Exercise 4 — Penalty kicks and minimax.

A striker chooses *Left* or *Right*; simultaneously, the goalkeeper chooses a side to dive to. The striker wants to maximise the following payoff, whereas the goalkeeper wants to minimise it:

$$A = \begin{pmatrix} 2 & 9 \\ 8 & 4 \end{pmatrix}.$$

Rows are the striker's choices and columns the goalkeeper's choices.

(a) Compute the lower and upper values when both players are restricted to pure strategies:

$$L = \max_i \min_j a_{ij}, \qquad H = \min_j \max_i a_{ij}.$$

Does the pure game have a value?

(b) Let the striker kick Left with probability p. Find the expected payoff against each pure action of the goalkeeper. Determine the value of p that maximises the striker's guaranteed payoff.

(c) Let the goalkeeper dive Left with probability q. Determine the choice of q that minimises the greatest payoff available to the striker.

(d) Verify directly that

$$p^* = \frac{4}{11}, \qquad q^* = \frac{5}{11}, \qquad v = \frac{64}{11}.$$

Show that p^* guarantees at least v against every mixed strategy of the goalkeeper, while q^* guarantees that the striker receives at most v against every mixed strategy of the striker. Explain why these two facts already establish the minimax equality for this game.

Exercise 5 — ★ Randomised algorithms against adversarial inputs.

Three deterministic algorithms A_1, A_2, A_3 can be run on three possible inputs I_1, I_2, I_3. Their costs are

$$C = \begin{array}{c|ccc} & I_1 & I_2 & I_3 \\ \hline A_1 & 0 & 3 & 3 \\ A_2 & 3 & 0 & 3 \\ A_3 & 3 & 3 & 0 \end{array}.$$

The algorithm designer wants to minimise cost; an adversary chooses the input.

(a) Suppose the adversary chooses inputs according to an arbitrary distribution $y = (y_1, y_2, y_3)$. Show that there is always a deterministic algorithm whose expected cost is at most 2.

(b) Find a randomised algorithm whose expected cost is at most 2 on every individual input. Is 2 optimal?

(c) Let $C = (c_{ij})$ now be an arbitrary finite cost matrix. Suppose that, for every probability distribution y over inputs, there exists a deterministic algorithm A_i such that

$$\sum_j y_j c_{ij} \le C_0.$$

Use the minimax theorem to prove that there exists a probability distribution x over algorithms such that, for every individual input I_j,

$$\sum_i x_i c_{ij} \le C_0.$$

(d) ★ Turn the statement around to obtain the following lower-bound principle:

> If there is a probability distribution over inputs under which every deterministic algorithm has expected cost at least C_0, then every randomised algorithm has some input on which its expected cost is at least C_0.

Explain carefully the order of the quantifiers. This is the finite matrix-game form of *Yao's minimax principle*.