How to write a translator to Dedukti

The case of Agda

Jesper Cockx
Thiago Felicissimo

25 June 2022

The goal of this talk

You have already seen how to define theories and write proofs in Dedukti (eg. the theory $\mathcal{U}$)

Now we will see how to write automatic translators from proof assistants to Dedukti

We will first discuss the general principles on writing such translator to Dedukti

We then discuss the specific case of the Agda2Dedukti translator

From Agda to Dedukti

1. Principles on translating from a proof assistant to Dedukti
2. Encoding Agda in Dedukti
3. Implementation of Agda2Dedukti
4. Inductive types and dependent pattern matching
5. Universe polymorphism
6. Eta equality & irrelevance
7. Conclusion

How to translate from a proof assistant to Dedukti

Step 0: Find (or define) a system $\mathcal{O}$ corresponding to the proof assistant's logic (not easy!)

Step 1: Define a Dedukti theory[1] $\mathcal{T}_{\mathcal{O}}$ representing the object logic in Dedukti, along with a translating function $[\![-]\!] : \Lambda_{\mathcal{O}} \to \Lambda_{DK}$.
The couple $(\mathcal{T}_{\mathcal{O}}, [\![-]\!])$ is an encoding of $\mathcal{O}$.

Step 2: Starting from the proof assistant code, implement the translating function

[1] Recall that a Dedukti theory is a pair $(\Sigma, \mathcal{R})$

Different levels of correctness

Not all encodings are equal!

An encoding is sound if

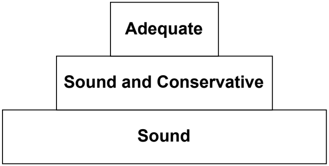
$$\vdash_{\mathcal{O}} M : A \quad \text{implies} \quad \vdash_{\mathcal{T}_{\mathcal{O}}} [\![M]\!] : \mathit{El}\ [\![A]\!]$$

An encoding is conservative if

$$\vdash_{\mathcal{T}_{\mathcal{O}}} M : \mathit{El}\ [\![A]\!] \quad \text{implies} \quad \exists N,\ \vdash_{\mathcal{O}} N : A$$

An encoding is adequate if for each type A,

$[\![-]\!]$ is a *compositional bijection* between A and $\mathit{El}\ [\![A]\!]$



Differences between core languages

Dependent types: Coq, Agda, Lean, ...

Inductive types: Most proof assistants
(type-theory assistants also have inductive families)

Universe polymorphism: Coq, Agda, Lean, ...

Impredicativity: All proof assistants, except Agda and Epigram

Eta-equality & irrelevance: Present in different levels in different proof assistants

Differences between implementations

Curry-Howard assistants (Coq/Agda/Matita): Proof terms are already in the internal syntax, easier to translate

LCF-like assistants (Isabelle/HOL): No proof terms, need to reconstruct them from proof derivations

Other cases:

- PVS: Proofs derivations are not even internally available... (see Gabriel's talk for a solution)
- ...

From Agda to Dedukti

What is Agda?

Agda is a dependently typed programming language and proof assistant based on Martin-Löf type theory.

It has indexed datatypes, dependent pattern matching, and explicit universe polymorphism.

Its type checker identifies terms up to β-equality and η-equality for functions and records, and supports definitional proof irrelevance.

Data types in Agda

```
data _⊎_ (A B : Set) : Set where
  left  : A → A ⊎ B
  right : B → A ⊎ B

data _≤_ : ℕ → ℕ → Set where
  ≤-zero : ∀ {n}               → zero  ≤ n
  ≤-suc  : ∀ {m n} → m ≤ n → suc m ≤ suc n
```

Pattern matching in Agda

$$\_{<}\_ : \mathbb{N} \to \mathbb{N} \to \mathsf{Set}$$
$$m < n = m \leq \mathsf{suc}\ n$$

$$\mathsf{compare} : (m\ n : \mathbb{N}) \to (m \leq n) \uplus (n < m)$$
$$\mathsf{compare\ zero}\quad n \quad = \mathsf{left}\ \ {\leq}\text{-}\mathsf{zero}$$
$$\mathsf{compare}\ (\mathsf{suc}\ m)\ \mathsf{zero} \quad = \mathsf{right}\ {\leq}\text{-}\mathsf{zero}$$
$$\mathsf{compare}\ (\mathsf{suc}\ m)\ (\mathsf{suc}\ n)\ \mathsf{with}\ \mathsf{compare}\ m\ n$$
$$\ldots \mid \mathsf{left}\ \ m{\leq}n \quad = \mathsf{left}\ \ ({\leq}\text{-}\mathsf{suc}\ m{\leq}n)$$
$$\ldots \mid \mathsf{right}\ n{<}m \quad = \mathsf{right}\ ({\leq}\text{-}\mathsf{suc}\ n{<}m)$$

Agda as a PTS

At its core, Agda is a pure type system with sorts Set ℓ where ℓ is a universe level.

$$\mathsf{U} : (\ell : \mathsf{Level}) \to \mathsf{Set}\ (\mathsf{lsuc}\ \ell)$$
$$\mathsf{U}\ \ell = \mathsf{Set}\ \ell$$

$$\begin{aligned}\mathsf{rule} :\ & (\ell_1\ \ell_2 : \mathsf{Level}) \\ & (A : \mathsf{Set}\ \ell_1)\ (B : A \to \mathsf{Set}\ \ell_2) \\ \to\ & \mathsf{Set}\ (\ell_1 \sqcup \ell_2)\end{aligned}$$
$$\mathsf{rule}\ \_\ \_\ A\ B = (x : A) \to B\ x$$

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$
Def. symbol	$[\![\mathsf{f}]\!]$	$=$
Constructor	$[\![\mathsf{D.c}]\!]$	$=$
Lambda	$[\![\lambda x \to u]\!]$	$=$
Application	$[\![u\ v]\!]$	$=$
Pi type	$[\![(x : A) \to B]\!]$	$=$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	`f`
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	`D__c`
Lambda	$[\![\lambda x \to u]\!]$	$=$	
Application	$[\![u\ v]\!]$	$=$	
Pi type	$[\![(x : A) \to B]\!]$	$=$	
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	`f`
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	`D__c`
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	`f`
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	`D__c`
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	???
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	???

Tarski- vs. Russell-style universes[2]

Agda uses Russell-style universes: Elements are *types* themselves.

$$\frac{A : \mathsf{Set}_l}{A \ \ \textsc{type}}$$

In Dedukti, if $A : \mathsf{Set}$, we cannot have $a : A$.
Thus, Dedukti uses a form of Tarski-style universes: Elements are *codes* that can be *interpreted* as types.

$$\frac{c : \mathtt{U}\ (\mathtt{set}\ l)}{\mathtt{El}\ (\mathtt{set}\ l)\ c \ \ \textsc{type}}$$

[2] https://www.cs.rhul.ac.uk/home/zhaohui/universes.pdf

Encoding Agda's PTS in Dedukti

```
Sort : Type.
set  : Lvl -> Sort.

U : (s : Sort) -> Type.
def El : (s : Sort) -> (a : U s) -> Type.

def axiom : Sort -> Sort.
[i] axiom (set i) --> set (s i).

def rule : Sort -> Sort -> Sort.
[i, j] rule (set i) (set j) --> set (max i j).
```

(We will see how to to define `Lvl` later.)

Encoding pi types

- Add a constant prod for encoding the pi type:

$$\frac{A : \mathtt{U}\ s_A \quad x : \mathtt{El}\ s_A\ A \vdash B : \mathtt{U}\ s_B}{\mathtt{prod}\ s_A\ s_B\ A\ B : \mathtt{U}\ (\mathtt{rule}\ s_A\ s_B)}$$

Encoding pi types

- Add a constant prod for encoding the pi type:

$$\frac{A : \mathtt{U}\ s_A \quad x : \mathtt{El}\ s_A\ A \vdash B : \mathtt{U}\ s_B}{\mathtt{prod}\ s_A\ s_B\ A\ B : \mathtt{U}\ (\mathtt{rule}\ s_A\ s_B)}$$

- Identify elements of prod with the *metatheoretic arrow type*:

$$\begin{aligned}&\mathtt{El}\ \_\ (\mathtt{prod}\ s_A\ s_B\ A\ B)\\ &\qquad = (x : \mathtt{El}\ s_A\ A) \to El\ s_B\ (B\ x)\end{aligned}$$

Encoding pi types in Dedukti

```
prod : (s_A : Sort) ->
       (s_B : Sort) ->
       (A : U s_A) ->
       (B : (El s_A A -> U s_B)) ->
       U (rule s_A s_B).

[s_A, s_B, A, B]
      El _ (prod s_A s_B A B)
  --> (x : El s_A A) -> El s_B (B x).
```

Reconstructing sorts

For translating pi types, we need access to the sort of the domain and codomain.

Luckily, Agda's type checker already annotates each type A with its sort $s(A)$.

Examples. $s(\mathbb{N}) = \mathsf{Set}$, $s(\mathsf{Set}) = \mathsf{Set}_1$,
$s(\mathsf{Set}_1 \rightarrow \mathsf{Set}) = \mathsf{Set}_2$

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	`f`
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	`D__c`
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	???
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	???

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	$\texttt{f}$
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	$\texttt{D\_\_c}$
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	$\texttt{prod}\ \lvert s(A)\rvert\ \lvert s(B)\rvert\ [\![A]\!]\ (x \Rightarrow [\![B]\!])$
	where $\lvert \mathsf{Set}\ \ell\rvert$	$=$	$\mathsf{set}\ [\![\ell]\!]$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	???

(We will see how to translate levels later.)

Encoding universes

- Add a constant $\mathtt{u}$ for encoding the Set type:

$$\frac{s : \mathtt{Sort}}{\mathtt{u}\ s : \mathtt{U}\ (\mathtt{axiom}\ s)}$$

Encoding universes

- Add a constant $\mathtt{u}$ for encoding the Set type:

$$\frac{s : \mathtt{Sort}}{\mathtt{u}\ s : \mathtt{U}\ (\mathtt{axiom}\ s)}$$

- Identify elements of $\mathtt{u}\ s$ with the ones of $\mathtt{U}\ s$:

$$\mathtt{El}\ \_\ (\mathtt{u}\ s) = \mathtt{U}\ s$$

In Dedukti:

```
u : (s : Sort) -> U (axiom s).
[i] El _ (u s) --> U s.
```

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	$\mathtt{f}$
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	$\mathtt{D\_\_c}$
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	$\mathtt{prod}\ \lvert s(A)\rvert\ \lvert s(B)\rvert\ [\![A]\!]\ (x \Rightarrow [\![B]\!])$
	where $\lvert\mathsf{Set}\ \ell\rvert$	$=$	$\mathsf{set}\ [\![\ell]\!]$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	???

(We will see how to translate levels later.)

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	$\mathtt{f}$
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	$\mathtt{D\_\_c}$
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	$\mathtt{prod}\ \lvert s(A) \rvert\ \lvert s(B) \rvert\ [\![A]\!]\ (x \Rightarrow [\![B]\!])$
	where $\lvert \mathsf{Set}\ \ell \rvert$	$=$	$\mathsf{set}\ [\![\ell]\!]$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	$\mathtt{u}\ (\mathsf{set}\ [\![\ell]\!])$

(We will see how to translate levels later.)

Encoding Agda definitions in Dedukti

Data types (no parameters or indices)

$$\left[\!\!\left[\begin{array}{l}\textsf{data}\ \mathsf{D} : U\ \textsf{where} \\ \quad \mathsf{c} : A\end{array}\right]\!\!\right] = \begin{array}{l}\texttt{D} : \texttt{El}\ |s(U)|\ [\![U]\!]\,. \\ \texttt{D\_\_c} : \texttt{El}\ |U|\ [\![A]\!]\,.\end{array}$$

Function definitions (no pattern matching)

$$\left[\!\!\left[\begin{array}{l}\mathsf{f} : A \\ \mathsf{f}\ x = v\end{array}\right]\!\!\right] = \begin{array}{l}\texttt{def f} : \texttt{El}\ |s(A)|\ [\![A]\!]\,. \\ \texttt{[x] f x -->}\ [\![v]\!]\,.\end{array}$$

From Agda to Dedukti

1. Principles on translating from a proof assistant to Dedukti
2. Encoding Agda in Dedukti
3. Implementation of Agda2Dedukti
4. Inductive types and dependent pattern matching
5. Universe polymorphism
6. Eta equality & irrelevance
7. Conclusion

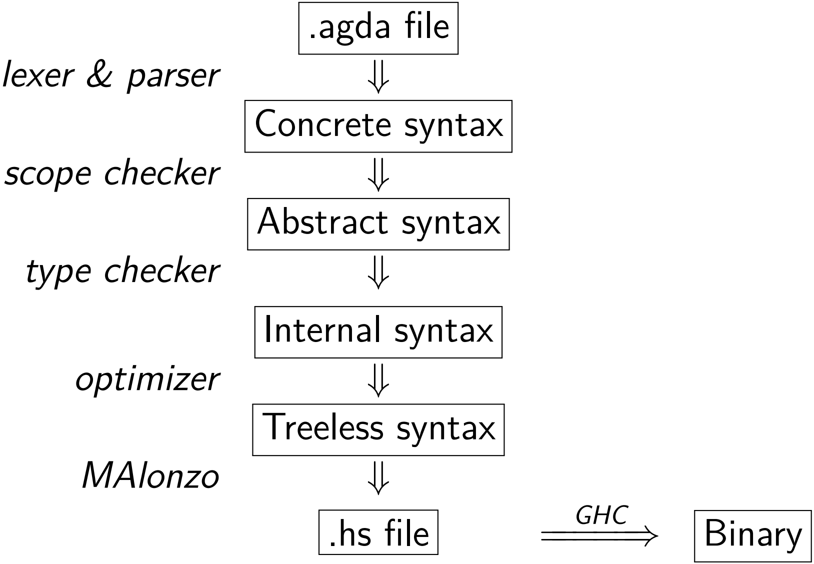
Implementation of Agda2Dedukti

Agda2Dedukti is implemented as an Agda backend.

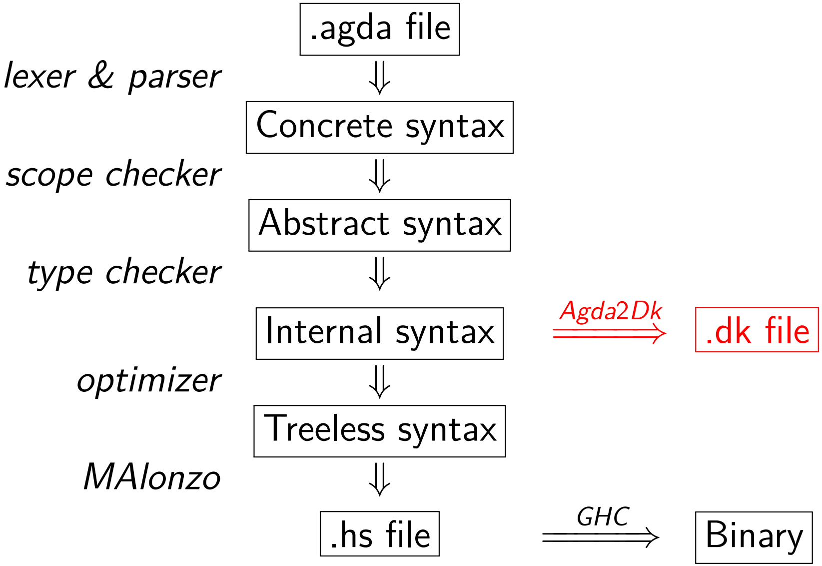
This allows us to reuse parts of Agda's implementation:

- Internal syntax representation
- Type checking monad `TCM`

Structure of the Agda typechecker



Structure of the Agda typechecker



Agda's internal syntax[3]

```
data Term
  = Var Int Elims                  -- x u v ..
  | Lam ArgInfo (Abs Term)         -- λ x → v
  | Lit Literal                    -- 42, 'a', ...
  | Def QName Elims                -- f u v ..
  | Con ConHead ConInfo Elims      -- c u v ..
  | Pi (Dom Type) (Abs Type)       -- (x : A) → B
  | Sort Sort                      -- Set, Set₁, Prop, ...
  | Level Level                    -- lzero, ...
  | MetaV MetaId Elims             -- _X_235
  | DontCare Term
  | Dummy String Elims
```

[3] Code from `Agda.Syntax.Internal`

Agda's TCM monad

Agda's typechecker uses a type-checking monad TCM:

```
type TCM a
getConstInfo :: QName -> TCM Definition
getBuiltin :: String -> TCM Term
getContext :: TCM Context
addContext :: (Name, Dom Type) -> TCM a -> TCM a
checkInternal :: Term -> Type -> TCM ()
reconstructParameters :: Type -> Term -> TCM Term
...
```

Putting it all together

```
example : (1 ≤ 2) ⊎ (2 < 1)
example = left (≤-suc ≤-zero)
```

```
{|!_⊎___left|}
  ({|!_≤_|}
    (Nat__suc Nat__zero)
    (Nat__suc (Nat__suc Nat__zero)))
  ({|!_<_|}
    (Nat__suc (Nat__suc Nat__zero))
    (Nat__suc Nat__zero))
  ({|!_≤___≤-suc|}
    Nat__zero
    (Nat__suc Nat__zero)
    ({|!_≤___≤-zero|} (Nat__suc Nat__zero)))
```

From Agda to Dedukti

1. Principles on translating from a proof assistant to Dedukti
2. Encoding Agda in Dedukti
3. Implementation of Agda2Dedukti
4. Inductive types and dependent pattern matching
5. Universe polymorphism
6. Eta equality & irrelevance
7. Conclusion

Translating datatypes and constructors to constants

Data types and their constructors do not reduce, so we translate them to constants in Dedukti.

Example. _≤_ is translated to:

```
{|!_≤_|} : El (set (s 0)) (prod (set 0) (set (s 0))
  Nat (_0 => (prod (set 0) (set (s 0))
    Nat (_0 => (u (set 0)))))).

{|!_≤___≤-zero|} : El (set 0) (prod (set 0) (set 0)
  Nat (n => ({|!_≤_|} Nat__zero n))).

{|!_≤___≤-suc|} : El (set 0) (prod (set 0) (set 0) Nat
  (m => (prod (set 0) (set 0)
    Nat (n => (prod (set 0) (set 0)
      ({|!_≤_|} m n)
      (_0 => ({|!_≤_|} (Nat__suc m) (Nat__suc n)))))))).
```

Reconstruction of data parameters

Constructors in Agda do *not* store their parameters.

Reconstructing parameters requires a type-directed traversal of the syntax.

We can reuse Agda's `reconstructParameters`, which does exactly this!

Filling implicit arguments & reconstructing parameters

$$\text{left}\ (\le\text{-suc}\ \le\text{-zero}) : (1 \le 2) \uplus (2 < 1)$$

Filling implicit arguments & reconstructing parameters

Agda's type checker infers implicit arguments during type checking.

$$\text{left } (\leq\text{-suc } \leq\text{-zero}) : (1 \leq 2) \uplus (2 < 1)$$
$$\Downarrow$$
$$\text{left } (\leq\text{-suc } \{m = 0\} \ \{n = 1\} \ (\leq\text{-zero } \{n = 1\}))$$

Filling implicit arguments & reconstructing parameters

Agda's type checker infers implicit arguments during type checking.

Agda2Dk makes all implicit arguments explicit and reconstructs constructor parameters.

$$\text{left } (\leq\text{-suc } \leq\text{-zero}) : (1 \leq 2) \uplus (2 < 1)$$

$$\Downarrow$$

$$\text{left } (\leq\text{-suc } \{m = 0\} \; \{n = 1\} \; (\leq\text{-zero } \{n = 1\}))$$

$$\Downarrow$$

$$\text{left } (1 \leq 2) \; (2 < 1) \; (\leq\text{-suc } 0 \; 1 \; (\leq\text{-zero } 1))$$

Translating clauses to rewrite rules

Functions in Agda are defined by a set of clauses, so we translate them to a constant + a set of rewrite rules.

Example. compare is translated to:

```
def compare : El (set 0) (prod (set 0) (set 0)
  Nat (m => (prod (set 0) (set 0)
    Nat (n => ({|!_⊎_|} ({|!_≤_|} m n) ({|!_<_|} n m)))))).
[n] compare Nat__zero n -->
  {|!_⊎___left|} ({|!_≤_|} Nat__zero n)
    ({|!_<_|} n Nat__zero) ({|!_≤___≤-zero|} n).
[m] compare (Nat__suc m) Nat__zero -->
  {|!_⊎___right|} ({|!_≤_|} (Nat__suc m) Nat__zero)
    ({|!_<_|} Nat__zero (Nat__suc m))
    ({|!_≤___≤-zero|} (Nat__suc (Nat__suc m))).
[m, n] compare (Nat__suc m) (Nat__suc n) -->
  {|!with-66|} m n (compare m n).
```

Drawbacks of generating rewrite rules

Generating a new rewrite rule for each clause means that we are extending the theory with each definition.

Moreover, checking correctness (completeness & termination) of rewrite rules is very hard.

Ongoing work: Instead, we can translate definitions by pattern matching to eliminators.[4]

[4] Ask Thiago for details!

From Agda to Dedukti

Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. *List Nat* but also *List* Set_0)

Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. *List Nat* but also $List\ Set_0$)

Bad solution One $List_i$ and one map_i for each univ i

Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. *List Nat* but also *List* Set_0)

Bad solution One $List_i$ and one map_i for each univ i

Universe polymorphism Allows definitions that can be used at multiple universe levels

Universe polymorphism

Sometimes one wishes to use a definition at multiple universes (e.g. *List Nat* but also *List* Set_0)

Bad solution One $List_i$ and one map_i for each univ i

Universe polymorphism Allows definitions that can be used at multiple universe levels

```
data List {i} (A : Set i) : Set i where
  [] : List A
  _::_ : A → List A → List A

map : {i j : Level} → {A : Set i} → {B : Set j}
  → (f : A → B) → List A → List B
map f [] = []
map f (x :: l) = f x :: map f l
```

Other ways of having universe polymorphism

Before going on, a comparison with another proof assistant you know.

	Coq	Agda
Typical ambiguity		
Cumulativity ($Set_i \subseteq Set_{i+1}$)		
Definitions carry constraints		

[5] For Coq's version, see Gaspard Ferey's PhD thesis

Other ways of having universe polymorphism

Before going on, a comparison with another proof assistant you know.

	Coq	Agda
Typical ambiguity	Yes	No
Cumulativity ($Set_i \subseteq Set_{i+1}$)		
Definitions carry constraints		

[5] For Coq's version, see Gaspard Ferey's PhD thesis

Other ways of having universe polymorphism

Before going on, a comparison with another proof assistant you know.

	Coq	Agda
Typical ambiguity	Yes	No
Cumulativity ($Set_i \subseteq Set_{i+1}$)		
Definitions carry constraints		

[5] For Coq's version, see Gaspard Ferey's PhD thesis

Other ways of having universe polymorphism

Before going on, a comparison with another proof assistant you know.

	Coq	Agda
Typical ambiguity	Yes	No
Cumulativity ($Set_i \subseteq Set_{i+1}$)	Yes	No
Definitions carry constraints		

[5] For Coq's version, see Gaspard Ferey's PhD thesis

Other ways of having universe polymorphism

Before going on, a comparison with another proof assistant you know.

	Coq	Agda
Typical ambiguity	Yes	No
Cumulativity ($Set_i \subseteq Set_{i+1}$)	Yes	No
Definitions carry constraints	Yes	No

[5] For Coq's version, see Gaspard Ferey's PhD thesis

Other ways of having universe polymorphism

Before going on, a comparison with another proof assistant you know.

	Coq	Agda
Typical ambiguity	Yes	No
Cumulativity ($Set_i \subseteq Set_{i+1}$)	Yes	No
Definitions carry constraints	Yes	No

Very different versions

In this talk we only see the encoding of Agda's universe polymorphism[5]

[5] For Coq's version, see Gaspard Ferey's PhD thesis

Universe polymorphism in Dedukti

Idea Generalize encoding of the arrow type

```
setOmega  : Sort.

forall : (l : (Lvl -> Sort)) ->
         ((i : Lvl) -> U (l i)) -> U setOmega.

[l, t] El _ (forall l t) -->
           (i : Lvl) -> El (l i) (t i).
```

Universe polymorphism in Dedukti

Idea Generalize encoding of the arrow type

```
setOmega  : Sort.

forall : (l : (Lvl -> Sort)) ->
         ((i : Lvl) -> U (l i)) -> U setOmega.

[l, t] El _ (forall l t) -->
          (i : Lvl) -> El (l i) (t i).
```

We extend the translation function with

Level quantification	$[\![(i : Level) \to A]\!]$	$=$	$\texttt{forall}\ (i \Rightarrow [\![s(A)]\!])\ (i \Rightarrow [\![A]\!])$
Level application	$[\![M\ l]\!]$	$=$	$[\![M]\!]\ [\![l]\!]$
Level abstraction	$[\![\lambda i.M]\!]$	$=$	$i \Rightarrow [\![M]\!]$

Back to List

Now the constant `List` can be given the type

```
El setOmega
   (forall (i => set (suc i))
           (i => prod (set (suc i))
                      (set (suc i))
                      (u (set i))
                      (_ => u (set i))))
```

Which, as expected, computes to

```
(i : Lvl) -> U (set i) -> U (set i)
```

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2\,.$$

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Idempotence $a \sqcup a = a$

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Idempotence $a \sqcup a = a$

Associativity $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Idempotence $a \sqcup a = a$

Associativity $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Commutativity $a \sqcup b = b \sqcup a$

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Idempotence $a \sqcup a = a$

Associativity $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Commutativity $a \sqcup b = b \sqcup a$

Distributivity $lsuc\ (a \sqcup b) = lsuc\ a \sqcup lsuc\ b$

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Idempotence $a \sqcup a = a$

Associativity $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Commutativity $a \sqcup b = b \sqcup a$

Distributivity $lsuc\ (a \sqcup b) = lsuc\ a \sqcup lsuc\ b$

Neutrality $a \sqcup lzero = a$

Universe levels

Levels are given by the syntax

$$l_1, l_2 ::= i \mid lzero \mid lsuc \mid l_1 \sqcup l_2 \,.$$

Levels are not freely generated, they satisfy:

Idempotence $a \sqcup a = a$

Associativity $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Commutativity $a \sqcup b = b \sqcup a$

Distributivity $lsuc\ (a \sqcup b) = lsuc\ a \sqcup lsuc\ b$

Neutrality $a \sqcup lzero = a$

Subsumption $a \sqcup lsuc^n\ a = lsuc^n\ a$

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$$l_1 \equiv l_2 \text{ should imply } [\![l_1]\!] \equiv [\![l_2]\!]$$

Possible solutions:

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$$l_1 \equiv l_2 \text{ should imply } [\![l_1]\!] \equiv [\![l_2]\!]$$

Possible solutions:

1. Representing levels as naturals?

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$$l_1 \equiv l_2 \text{ should imply } [\![l_1]\!] \equiv [\![l_2]\!]$$

Possible solutions:

1. Representing levels as naturals? *Closed terms do not satisfy all equalities (e.g.* $i \sqcup j \not\equiv j \sqcup i$*).*

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$$l_1 \equiv l_2 \text{ should imply } [\![l_1]\!] \equiv [\![l_2]\!]$$

Possible solutions:

1. Representing levels as naturals? *Closed terms do not satisfy all equalities (e.g. $i \sqcup j \not\equiv j \sqcup i$).*
2. Representing levels as a set of variables with natural increments? **(current solution)**

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$l_1 \equiv l_2$ should imply $[\![l_1]\!] \equiv [\![l_2]\!]$

Possible solutions:

1. Representing levels as naturals? *Closed terms do not satisfy all equalities (e.g. $i \sqcup j \not\equiv j \sqcup i$).*
2. Representing levels as a set of variables with natural increments? **(current solution)** *Works well, but there is a catch (next slide).*

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$$l_1 \equiv l_2 \text{ should imply } [\![l_1]\!] \equiv [\![l_2]\!]$$

Possible solutions:

1. Representing levels as naturals? *Closed terms do not satisfy all equalities (e.g. $i \sqcup j \not\equiv j \sqcup i$).*
2. Representing levels as a set of variables with natural increments? **(current solution)** *Works well, but there is a catch (next slide).*
3. Decision procedure integrated in Dedukti?

The challenge of representing universe polymorphism

To establish the encoding's soundness,

$$l_1 \equiv l_2 \text{ should imply } [\![l_1]\!] \equiv [\![l_2]\!]$$

Possible solutions:

1. Representing levels as naturals? *Closed terms do not satisfy all equalities (e.g. $i \sqcup j \not\equiv j \sqcup i$).*
2. Representing levels as a set of variables with natural increments? **(current solution)** *Works well, but there is a catch (next slide).*
3. Decision procedure integrated in Dedukti? *We leave this to the future generations.*

Current solution: levels as sets

Idea. Every level l admits a unique canonical form

$$l = \max\{n, i_1 + m_1, ..., i_k + m_k\}$$

where $i_1, .., i_k \in FV(l)$, $n, m_1, .., m_k \in \mathbb{N}$ and $m_j \leq n$.

Current solution: levels as sets

Idea. Every level l admits a unique canonical form

$$l = \max\{n, i_1 + m_1, ..., i_k + m_k\}$$

where $i_1, .., i_k \in FV(l)$, $n, m_1, .., m_k \in \mathbb{N}$ and $m_j \leq n$.

A rewrite system can calculate such forms by using rewriting modulo associativity-commutativity.

Current solution: levels as sets

Idea. Every level l admits a unique canonical form

$$l = \max\{n, i_1 + m_1, ..., i_k + m_k\}$$

where $i_1, .., i_k \in FV(l)$, $n, m_1, .., m_k \in \mathbb{N}$ and $m_j \leq n$.

A rewrite system can calculate such forms by using rewriting modulo associativity-commutativity.

But idempotence and subsumption require a non-linear rule:

$$\max\{i + n, i + m\} = i + \max\{n, m\}$$

Current solution: levels as sets

Idea. Every level l admits a unique canonical form

$$l = \max\{n, i_1 + m_1, ..., i_k + m_k\}$$

where $i_1, .., i_k \in FV(l)$, $n, m_1, .., m_k \in \mathbb{N}$ and $m_j \leq n$.

A rewrite system can calculate such forms by using rewriting modulo associativity-commutativity.

But idempotence and subsumption require a non-linear rule:

$$\max\{i + n, i + m\} = i + \max\{n, m\}$$

This breaks confluence of pre-terms, and prevents proving conservativity without changing Dedukti.

From Agda to Dedukti

Eta equality in Agda

Agda supports two kinds of eta-equality:

1. Eta for functions:

$$\frac{f : (x : A) \to B}{f = (\lambda x \to f\, x) : (x : A) \to B}$$

2. Eta for records:[6]

$$\frac{u : \Sigma\ A\ B}{u = (\mathsf{proj}_1\ u, \mathsf{proj}_2\ u) : \Sigma\ A\ B}$$

[6] Also known as surjective pairing for Σ.

Definitional singleton types

Agda supports eta for *all* record types, not just Σ!
In particular, it has eta for the unit type:

```
record ⊤ : Set where -- no fields
  constructor tt

eta-unit : (x y : ⊤) → x ≡ y
eta-unit x y = refl
```

Two distinct variables might be equal!

⇒ To check if two terms are convertible, it does not suffice to compare their normal forms.

Encoding eta in Dedukti

1. Eta-expand everything when translating?

Encoding eta in Dedukti

1. Eta-expand everything when translating?
   This is not stable under substitution:

   $$(\lambda a : A.a)\{Nat \to Nat/A\}$$

   is not in eta-long form, but $\lambda a : A.a$ and $Nat \to Nat$ are.

Encoding eta in Dedukti

1. Eta-expand everything when translating?
   This is not stable under substitution:

   $$(\lambda a : A.a)\{Nat \to Nat/A\}$$

   is not in eta-long form, but $\lambda a : A.a$ *and* $Nat \to Nat$ *are.*
2. Eta-reduce everything when translating?

Encoding eta in Dedukti

1. Eta-expand everything when translating?
   This is not stable under substitution:

   $$(\lambda a : A.a)\{Nat \to Nat/A\}$$

   is not in eta-long form, but $\lambda a : A.a$ *and* $Nat \to Nat$ *are.*
2. Eta-reduce everything when translating?
   This is not stable under substitution and β*:*

   $$(\lambda x.y\ x\ x)\{(\lambda x'.z)/y\} \hookrightarrow_\beta \lambda x.z\ x \hookrightarrow_\eta z$$

   but $\lambda x.y\ x\ x \not\hookrightarrow_\eta$ *and* $\lambda x'.z \not\hookrightarrow_\eta$.

Encoding eta in Dedukti

3. Add eta-equality to the metatheory?

Encoding eta in Dedukti

3. Add eta-equality to the metatheory?
   This only handles eta for the arrow type.

Encoding eta in Dedukti

3. Add eta-equality to the metatheory?
   This only handles eta for the arrow type.

4. Use eta-reduction for record types?

Encoding eta in Dedukti

3. Add eta-equality to the metatheory?
   This only handles eta for the arrow type.

4. Use eta-reduction for record types?
   This does not work for unit type, and needs non-linearity for the others:

   ```
   mk_pair (pi_1 p) (pi_2 p) --> p
   ```

Encoding eta in Dedukti

3. Add eta-equality to the metatheory?
   This only handles eta for the arrow type.

4. Use eta-reduction for record types?
   This does not work for unit type, and needs non-linearity for the others:
   ```
   mk_pair (pi_1 p) (pi_2 p) --> p
   ```

5. Annotate terms with their types to be able to match them to eta expand? e.g.
   ```
   eta (arrow nat nat) f --> x => f x
   ```

Encoding eta in Dedukti

3. Add eta-equality to the metatheory?
   This only handles eta for the arrow type.

4. Use eta-reduction for record types?
   This does not work for unit type, and needs non-linearity for the others:
   `mk_pair (pi_1 p) (pi_2 p) --> p`

5. Annotate terms with their types to be able to match them to eta expand? e.g.
   `eta (arrow nat nat) f --> x => f x`
   We get huge terms, and the other rules make the system non-confluent on pre-terms.

Encoding eta in Dedukti

The next idea. Extend Dedukti with typed-directed rewrite rules.

Take inspiration from already existing works:

- Agda's implementation of eta
- Andromeda 2's extensionality rules

Or maybe there are still other unexplored options?

Definitional irrelevance

Agda also supports definitional proof irrelevance [7] for irrelevant functions and elements of Prop:

```
postulate
  P : Prop
  f : P → ℕ

P-irrelevant : (x y : P) → f x ≡ f y
P-irrelevant x y = refl
```

This causes very similar problems to eta for ⊤, that also requires type-directed conversion to solve.

[7] In PVS we have a simpler form of proof irrelevance, which can be encoded in Dedukti.

From Agda to Dedukti

1. Principles on translating from a proof assistant to Dedukti
2. Encoding Agda in Dedukti
3. Implementation of Agda2Dedukti
4. Inductive types and dependent pattern matching
5. Universe polymorphism
6. Eta equality & irrelevance
7. Conclusion

Summary

Many features of a dependently typed language can be encoded in Dedukti directly:

- Defined symbols are mapped to constants.
- Clauses are mapped to rewrite rules.

Other features require some more work:

- Erased constructor parameters need to be reconstructed.
- Universe levels require an equational theory.

Finally, other features we don't yet know how to encode:

- Eta-equality for record types?
- Definitional proof irrelevance?

Future work

Like most translators, Agda2Dedukti is a WIP

In the future, we would like to have

- Compilation of clauses to elimination principles
- A conservative encoding of universe polymorphism
- Adequate and computational encoding of Agda[8]
- An encoding of eta-equality and irrelevance (probably needs to extend Dedukti)

[8] For details, see Thiago's talk about Adequate and Computational Encodings in Dedukti, at FSCD 2022

References

- G. Genestier. Encoding Agda Programs Using Rewriting. In Proceedings of the 5th International Conference on Formal Structures for Computation and Deduction, Leibniz International Proceedings in Informatics 167, 2020.[9]
- T. Felicissimo. Representing Agda and coinduction in the lambda-pi calculus modulo rewriting. Master thesis, 2021.[10]

[9] https://drops.dagstuhl.de/opus/volltexte/2020/12353/pdf/LIPIcs-FSCD-2020-31.pdf

[10] https://hal.inria.fr/hal-03343699