Logique
TD n°9bis

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1 Constructive proofs

We write $\Gamma \vdash_c A$ if $\Gamma \vdash A$ in classical natural deduction and $\Gamma \vdash_i A$ if $\Gamma \vdash A$ in constructive (the "i" stands for *intuitionistic*).

Exercise 1 : Constructive proofs and negations

Prove that $\vdash_i A \Rightarrow \neg\neg A$ and $\vdash_i \neg\neg\neg A \Rightarrow \neg A$. Draw the graph of vertices $\neg^k A$ for $k \in \mathbb{N}$ and of edges implications that are constructively provable. Which edge(s) are added in the classical case?

Exercise 2 : Constructive properties

1. Prove that $A \vee \neg A$ is not provable in constructive natural deduction for all formulas A.
2. Prove that if $\vdash_i \exists x.\ A$, then there is a term t such that $\vdash_i (x/t)A$.
3. Prove that if $\vdash_i \forall x \exists y.\ A$, then there is a function f from closed terms to terms such that for every term t, $\vdash_i (y/f(t), x/t)A$.
4. Prove that the formula $\exists x. (P(x) \Rightarrow \forall y. P(y))$ is not provable in constructive natural deduction.

Exercise 3 : Double negation translation

Given a formula A, we define its GÖDEL translation (or double negation translation) by structural induction over A:

- if A is atomic, $\mathcal{G}(A) = \neg\neg A$
- $\mathcal{G}(\top) = \top$
- $\mathcal{G}(\bot) = \bot$
- $\mathcal{G}(\neg A) = \neg\mathcal{G}(A)$
- $\mathcal{G}(A_1 \wedge A_2) = \mathcal{G}(A_1) \wedge \mathcal{G}(A_2)$
- $\mathcal{G}(A_1 \vee A_2) = \neg\neg(\mathcal{G}(A_1) \vee \mathcal{G}(A_2))$

- $\mathcal{G}(A_1 \Rightarrow A_2) = \mathcal{G}(A_1) \Rightarrow \mathcal{G}(A_2)$
- $\mathcal{G}(\forall x.\, A) = \forall x.\, \mathcal{G}(A)$
- $\mathcal{G}(\exists x.\, A) = \neg\neg\exists x.\, \mathcal{G}(A)$

If Γ is a set of formulas, we write $\mathcal{G}(\Gamma)$ the set $\{\mathcal{G}(A) \mid A \in \Gamma\}$. The goal of this exercise is to prove that $\Gamma \vdash_c A$ iff $\mathcal{G}(\Gamma) \vdash_i \mathcal{G}(A)$.

1. Prove that for every formula A, $\vdash_i \neg\neg\mathcal{G}(A) \Rightarrow \mathcal{G}(A)$. You will at least treat the cases of $\neg$; $\wedge$; $\exists$.
   You can use that $\neg\neg(A \wedge B) \Rightarrow (\neg\neg A \wedge \neg\neg B)$ *is constructively provable.*
2. Show that for every formula A, if $\Gamma \vdash_c A$ then $\mathcal{G}(\Gamma) \vdash_i \mathcal{G}(A)$. You will at least treat the cases of: introduction of $\wedge$ and $\vee$; elimination of $\vee$; RAA.
3. Prove that for every formula A, $\vdash_c A \Leftrightarrow \mathcal{G}(A)$.
4. Show that for every formula A, if $\mathcal{G}(\Gamma) \vdash_i \mathcal{G}(A)$ then $\Gamma \vdash_c A$.

2 Kripke structures and constructive logic

Let $\mathcal{K}$ be a Kripke model with worlds W and order $\leq$. Let ϕ be a valuation. We will write $\mathcal{K}, w, \phi \models A$ for $[\![\phi]\!]^w_\phi = 1$. When the context is clear, we may omit the structure and valuation in this notation.

Exercise 4 : Kripke structures

In the following, X, Y, Z are constant predicates.

1. Is the structure:

$$\begin{tikzcd}
X, Z \bullet w_1 & & w_2 \bullet X, Y \\
& w_0 \bullet \; X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

   a model of $(\neg Y \wedge X) \Rightarrow Z$?
2. Let $\mathcal{K}$ be a Kripke structure with worlds W and order $\leq$, and $w \in W$ a world.
   (a) What does $\mathcal{K}, w \models \neg\neg X$ mean?
   (b) What does $\mathcal{K}, w \models \neg(\neg X \wedge \neg Y)$ mean?
3. Give Kripke counter-models for the following formulas if they are not provable (if they are provable, no need for a full proof):
   (a) $X \Rightarrow \neg\neg X$
   (b) $\neg\neg X \Rightarrow X$
   (c) $\neg X \vee \neg\neg X$
   (d) $\neg(X \wedge Y) \Rightarrow (\neg X \vee \neg Y)$
   (e) $(\neg X \vee \neg Y) \Rightarrow \neg(Y \wedge X)$
   (f) $(X \Rightarrow Y) \Rightarrow (\neg X \vee Y)$
   (g) $\neg\neg(A \wedge B) \Leftrightarrow (\neg\neg A \wedge \neg\neg B)$
   (h) $\neg\neg(A \Rightarrow B) \Leftrightarrow (\neg\neg A \Rightarrow \neg\neg B)$

4. Is the Kripke structure

$$\begin{tikzcd} \bullet\ \mathbb{R} \\ \bullet\ \mathbb{N} \arrow[u] \end{tikzcd}$$

a model of $\exists x.\, \exists y.\, x < y \wedge \neg(\exists z. x < z \wedge z < y)$?

Exercise 5 : Independent connectives

We say that a binary connective $\otimes$ is independent from a set of connectives C is there is no formula A using X and Y and built only with connectives from C such that $\vdash_i (X \otimes Y) \Leftrightarrow A$.

1. Show that if $\vee$ is not independent from $\{\bot, \wedge, \neg, \Rightarrow\}$, then $\vdash_i \neg\neg(X \vee Y) \Leftrightarrow (\neg\neg X \vee \neg\neg Y)$. *Hint: use Exercise 4, questions 3. (g) and (h).* Conclude.

We consider the Kripke structure $\mathcal{K}$ of worlds $W = \{\omega_1, \omega_2, \omega_3\}$ with $\omega_1 \leq \omega_3$, $\omega_2 \leq \omega_3$ and $\hat{X}^{\omega_1} = \hat{Y}^{\omega_2} = \hat{X}^{\omega_3} = \hat{Y}^{\omega_3} = 1$ and $\hat{Y}^{\omega_1} = \hat{X}^{\omega_2} = 0$.

Advice: draw the associated Kripke structure

2. Show that for all proposition A containing only X, Y, $\bot$, $\neg$, $\vee$ and $\Rightarrow$, if $\omega_3 \models A$ then $\omega_1 \models A$ or $\omega_2 \models A$. Conclude that $\wedge$ is independent from $\{\bot, \vee, \neg, \Rightarrow\}$.

Exercise 6 : Excluded middle

We denote by $\mathcal{P}_0$ the set of nullary predicates in our language, which we assume nonempty. Let $\mathcal{K}$ be the Kripke structure of worlds the *partial interpretations*, i.e. the ordered pairs (I, f) where $I \subseteq \mathcal{P}_0$ and $f : I \longrightarrow \{0, 1\}$. For every $X \in \mathcal{P}_0$, $\hat{X}^{(I,f)} = 1$ iff $X \in I$ and $f(X) = 1$.

The structure $\mathcal{K}$ is ordered by $\sqsubseteq$, where $(I, f) \sqsubseteq (J, g)$ iff $I \subseteq J$ and for every $X \in I$, $f(X) = g(X)$.

1. What does $\mathcal{K}, (I, f) \models \neg X$ mean, where $X \in I$?
2. Show that $\mathcal{K}$ is a counter-model to $X \vee \neg X$.
3. Give a formula A which is not provable constructively, but which is satisfied by the structure $\mathcal{K}$.

Additional exercises on constructive logic

Exercise 7 :

Let A be a propositional formula, provable in classical logic (a tautology). We define F_2 the ordered set $\omega_1 \leq \omega_2$. We call structure of base F_2 every Kripke structure of underlying ordered set F_2. We call $\mathsf{LI} + A$ the set of formulas provable when adding to natural deduction the following rule:

$$\frac{}{\vdash (X_1/B_1, \ldots, X_n/B_n)A}$$

For example, $\mathsf{LI} + (X \vee \neg X)$ is LC, the set of formulas provable in classical logic.

The goal of this exercise is to prove YANKOV's theorem : *for every tautology A, $\mathsf{LI} + A = \mathsf{LC}$ iff A is not satisfied in a structure of base F_2.*

1. Show that if $\mathsf{LI} + A = \mathsf{LC}$ then A is not satisfied in a structure of base F_2. *Hint: you can use the counter-model to $\neg\neg X \Rightarrow X$ found in Exercise 1.*
2. Let $\mathcal{K}$ be the Kripke structure of underlying set F_2 such that $\hat{X}^{w_2} = 1$ and $\hat{X}^{w_1} = 0$. Assume that A has a single propositional variable X. Show that if A is not satisfied in $\mathcal{K}$ then every structure $\mathcal{K}'$ of underlying set W' satisfying A is such that for every world $w \in W'$, $\hat{X}^w = 0$ implies that there is a world $w' \geq w$ such that for every world $w'' \geq w'$, $\hat{X}^{w''} = 0$.
3. Conclude: if A is a formula with a single propositional variable X and $\mathcal{K}$ does not satisfy A, then $A \vdash_i \neg\neg X \Rightarrow X$.
4. Let A a proposition of propositional variables $X_1, \ldots, X_n$. Show that if A is not satisfied by a structure of base F_2, then there are formulas $B_1, \ldots, B_n$ with one propositional variable X and such that $\mathcal{K}$ does not satisfy $(X_1/B_1, \ldots, X_n/B_n)A$.
5. Conclude.
6. Show that if $A_1, \ldots, A_n$ are tautologies and $\mathsf{LI} + A_1 \wedge \ldots \wedge A_n = \mathsf{LC}$, then there is $i \in \{1, \ldots, n\}$ such that $\mathsf{LI} + A_i = \mathsf{LC}$.

Exercise 8 :

We call Heyting arithmetic the constructive theory of axioms those of Peano arithmetic, i.e. its theorems are all formulas provable in constructive logic from the Peano axioms. We write $HA \vdash_i A$ when A is a theorem of Heyting arithmetic.

1. Show that equality is decidable in Heyting arithmetic:

$$HA \vdash_i \forall x. \forall y. (x = y \vee x \neq y)$$

   Hint: use induction and $\forall x.(x = 0 \vee \exists y.\ x = S(y))$.

2. The goal is to show that Heyting arithmetic has the witness property, i.e. if $HA \vdash_i \exists x. A$, then there is $n \in \mathbb{N}$ such that $HA \vdash_i (x/\underline{n})A$, where $\underline{n}$ is the term $S^n(0) = S(\ldots(S(0)))$. By contradiction, let us assume that for every natural number n, $HA \nvdash_i (x/\underline{n})A$. Then, for every n, there is a Kripke structure $\mathcal{K}_n$ such that $\mathcal{K}_n$ satisfies all Peano axioms but not $(x/\underline{n})A$. We built the structure $\mathcal{K} = \{\omega\} \sqcup \bigsqcup_{n\in\mathbb{N}} \mathcal{K}_n$ with smallest element ω, in which $\mathcal{D}_\omega = \mathbb{N}$, S is interpreted as the successor, 0 as 0, $+$ as the addition, $\times$ as the multiplication and $=$ as the equality.
   (a) Show that $\mathcal{K}$ is a Kripke structure.
   (b) Show that $\mathcal{K}, \omega \not\models \exists x. A$.
   (c) Show that $\mathcal{K}$ satisfies all Peano axioms (you can restrict to the induction scheme).
3. We want to show that if $HA \vdash_i A \vee B$ then $HA \vdash_i A$ or $HA \vdash_i B$.

(a) Show that for all formulas A and B not containing the variable x,

$$HA \vdash_i (A \vee B) \Leftrightarrow \exists x. (x = 0 \Rightarrow A) \wedge (x \neq 0 \Rightarrow B)$$

(b) Conclude.

Exercise 9 : A topological semantic

We restrict ourselves to the propositional fragment of constructive logic, i.e. we consider only quantifier-free formulas. We give a first semantic to this logic, the topological semantics given by TARSKI.

A topological space is defined by a set E and a set $\mathcal{O} \subseteq \mathfrak{P}(E)$ such that:

- the empty set $\varnothing$ is in $\mathcal{O}$
- E is in $\mathcal{O}$
- if $(U_i)_{i\in I}$ is a family of $\mathcal{O}$, then $\bigcup_{i\in I} U_i \in \mathcal{O}$
- if $(U_i)_{i\in I}$ is a finite family of $\mathcal{O}$, $\bigcap_{i\in I} U_i \in \mathcal{O}$

Elements of $\mathcal{O}$ are said to be *open*. Given a subset $W \subseteq E$, we define:

- $c(W) = E \setminus W$, the complement of W
- $i(W)$, the largest open set included in W, called the *interior* of W

A topological interpretation is defined by a topological space $\langle E, \mathcal{O}\rangle$ and a function σ from variables to $\mathcal{O}$. This interpretation is extended to all formulas A by structural induction:

- $[\![\top]\!] = E$
- $[\![\bot]\!] = \varnothing$
- $[\![X]\!] = \sigma(X)$
- $[\![A_1 \wedge A_2]\!] = [\![A_1]\!] \cap [\![A_2]\!]$
- $[\![A_1 \vee A_2]\!] = [\![A_1]\!] \cup [\![A_2]\!]$
- $[\![\neg A]\!] = i(c([\![A]\!]))$
- $[\![A_1 \Rightarrow A_2]\!] = i(c([\![A_1]\!]) \cup [\![A_2]\!])$

Note that $[\![A]\!]$ is open. We call $[\![\Gamma]\!]$ the open set $\bigcap_{A\in\Gamma} [\![A]\!]$.

1. Prove that if $\Gamma \vdash A$ is provable in constructive logic, then $[\![\Gamma]\!] \subseteq [\![A]\!]$ for every topological interpretation.
2. Give formulas that are not provable in constructive logic (excluded-middle, de Morgan formulas,...). You can use $\mathbb{R}$ with its usual topology (generated by open intervals).

Remark: the converse to question 1 is true, but harder to prove.