Logique
TD quotients

Luc Chabassier
chabassier@lsv.fr
Amélie Ledein
ledein@lsv.fr

March 9, 2022

Some exercises on quotients.

A *equivalence relation* on a set S is a set $R \subseteq S \times S$ such that :

- $\forall x \in S, (x, x) \in R$ (reflexivity)
- $\forall x, y \in S, (x, y) \in R \implies (y, x) \in R$ (symmetry)
- $\forall x, y, z \in S, (x, y) \in R \land (y, z) \in R \implies (x, z) \in R$ (transitivity)

A *partition* P on a set S is a family of parts of S (ie $P \subseteq \mathscr{P}(S)$) such that :

- $\forall X \in P, X \neq \emptyset$
- $\forall X, Y \in P, X \neq Y \implies X \cap Y = \emptyset$
- $S = \bigcup_{X \in P} X$

Exercise 1 : Partitions and equivalences

Show that there is a bijection between partitions and equivalence relationships on a set S.

Given a set S and an equivalence relation R on it, we note S/R the set of parts that constitutes the partition associated to R by the previous bijection. This is called the *quotient* of S by R.

Exercise 2 : Quotient map

Show that there is a surjective map $q : S \to S/R$ such that $q(x) = q(y)$ if and only if $(x, y) \in R$.

Such a map is called a *quotient map*. $q(x)$ is often written $[x]$.

Exercise 3 : Lifting

1. Let $f : S \to X$. Show that f can factor through q if and only if $(x, y) \in R \implies f(x) = f(y)$.
2. Let $f : S \to X$. Let $f_1, f_2 : S/R \to X$ such that $f = f_1 \circ q = f_2 \circ q$. Show that $f_1 = f_2$.
3. Let $f : S_1 \to S_2$. Let R_1 and R_2 be equivalence relations respectively on S_1 and S_2. Show that there is a function $[f] : S_1/R_1 \to S_2/R_2$ such that $q_2 \circ f = [f] \circ q_1$ if and only if $(x, y) \in R_1 \implies (f(x), f(y)) \in R_2$.

Exercise 4 : Universal characterisation

Let $m : S \to M$ such that :

- $\forall x, y \in S, (x, y) \in R \implies m(x) = m(y)$
- $\forall f : S \to X, (\forall x, y \in S, (x, y) \in R \implies f(x) = f(y)) \implies \exists! f' : M \to X, f = f' \circ m$

Show that there is a unique bijection $i : M \approx S/R$ such that $i \circ m = q$.

Exercise 5 : Quotient on arbitrary relation

Let $R \subseteq S \times S$ be an **arbitrary** relation on S.

1. Show that one can define S/R and $q : S \to S/R$ such that it has the same properties as m in exercises 4.
2. Show that there is an equivalence relation R^* on S such that $R \subseteq R^*$ and for all equivalence relations R' containing R, $R^* \subseteq R'$.
3. Show that for any relation R' on S such that $R \subseteq R' \subseteq R^*$, $S/R = S/R'$ and the quotient maps are the same.

In particular notice that $S/R = S/R^*$.

Exercise 6 : Optional, for the category theoretists

Define the quotient as a coequalizer in *Set*.