Langages Formels

TD 2

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Exercise 1 : Des étoiles

On considère les énoncés suivants, de plus en plus forts.

Soit $L \subseteq \Sigma^*$, il existe $N > 0$ tel que pour tout $x \in L$:

(a) Si $|x| \geq N$ alors il peut s'écrire $x = u_1u_2u_3$ avec $u_2 \neq \varepsilon$ et $u_1u_2^*u_3 \subseteq L$.

(b) Si $x = w_1w_2w_3$ avec $|w_2| \geq N$ alors $w_2 = u_1u_2u_3$ avec $u_2 \neq \varepsilon$ et $w_1u_1u_2^*u_3w_3 \subseteq L$.

(c) Si $x = uv_1v_2 \ldots v_Mw$ avec $|v_i| \geq 1$ et $M \geq N$, alors il existe $0 \leq j < k \leq M$ tel que $uv_1 \ldots v_j(v_{j+1} \ldots v_k)^*v_{k+1} \ldots v_Mw \subseteq L$.

1. Montrer que tout langage reconnaissable vérifie l'énoncé (a).
2. Montrer que $\{ \ a^nb^n \ : \ n \in \mathbb{N} \ \}$ n'est pas reconnaissable.

On admet que les langages reconnaissables satisfont les trois propriétés (a), (b), et (c).

3. Montrer que le langage $\{ \ a^{n^2} \ : \ n \in \mathbb{N} \ \}$ n'est pas reconnaissable.
4. Montrer que $\{ \ u \ : \ u = \widetilde{u} \ \}$, c'est à dire le langage des palindromes, n'est pas reconnaissable.
5. De même pour $\{ \ (ab)^n(cd)^n \ : \ n \in \mathbb{N} \ \} \cup (\Sigma^*\{ \ aa, bb, cc, dd, ac \ \}\Sigma^*)$.
6. Montrer que le langage suivant satisfait (c) mais n'est pas reconnaissable : $\{ \ udv \ : \ u, v \in \{a, b, c\}^*, \ u \neq v$ ou bien l'un des deux contient un carré $\}$. On pourra admettre l'existence d'une infinité de mots sans carrés sur l'alphabet $\{a, b, c\}$.

Exercise 2 : Brzozowski-McCluskey algorithm

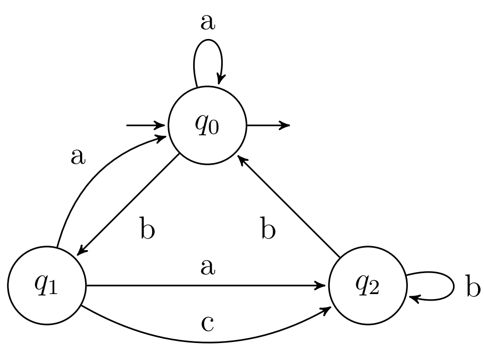
The goal of this exercise is to translate a finite automaton into a rational expression, giving an alternate proof of the associated implication of Kleene's theorem. We will proceed by successive transformations of the automaton.

1. We call *strongly normalized* every automaton which has a unique initial state to which no transition leads and a unique final state with no exiting transition, i.e. an automaton $\mathcal{A} = \langle Q, \Sigma, \{i\}, \{f\}, \delta \rangle$ such that for every state q and letter a, $(q, a, i) \notin \delta$ and $(f, a, q) \notin \delta$. Show that for all finite automaton, there is a strongly normalized automaton which recognizes the same language.

We will use a generalization of the definition of finite automata : the transition function will be a subset of $Q \times 2^{\Sigma^*} \times Q$. An execution of such an automaton recognizes the concatenation of languages of the transitions' labels. The automaton recognizes the union of the languages of all its accepting executions.

2. Show that every generalized automaton is equivalent to a generalized automaton in which there exists exactly one transition between each pair of states : $q' \in \delta(q, L)$ et $q' \in \delta(q, L')$ implies $L = L'$.

3. Let $\mathcal{A}$ be a strongly normalized generalized automaton with initial state i and final state f. Let $q \in Q_{\mathcal{A}}$, $q \notin \{i, f\}$. Show that there exists an automaton equivalent to $\mathcal{A}$ with set of states $Q_{\mathcal{A}} \setminus \{q\}$.
4. Conclude that if L is recognized by a strongly normalized generalized automaton $\mathcal{A}$, then L belongs to the rational closure of the labels of the transitions of $\mathcal{A}$.
5. Show that every finite automaton has an equivalent generalized automaton.
6. Give a procedure which, given a finite automaton, outputs a rational expression of same language.
7. Apply the construction to compute a rational expression corresponding to the following automaton :



Exercise 3 : Minimization by Brzozowski inversion

1. Show that the determinized of a co-deterministic co-accessible automaton which recognizes a language L is (isomorphic to) the minimal automaton of L.
2. Using this result, devise a procedure to minimize an automaton. What is the complexity of this method ?

Exercise 4 : Blind bartender

A blind bartender with boxer gloves plays the following game with a customer : he has a tray in front of him with four glasses arranged in a square. Each of these glasses may or may not be turned, without the bartender knowing. The purpose of the bartender is to arrange so that all the glasses are turned in the same direction. To do this, he can each turn choose one of the following three actions :

— Turn one of the glasses
— Turn two neighboring glasses
— Turn two opposing glasses

To add to the difficulty, the customer can turn the tray any number of quarter turns between each of the bartender's actions. The game ends as soon as one of the two winning positions is reached.

Show that we can restrict the number of different configurations to four, then represent the possible actions of the game by a non-deterministic automaton.

Determine this automaton and deduce a winning strategy for the bartender.

Contrôle continu à rendre le 12 février 2026

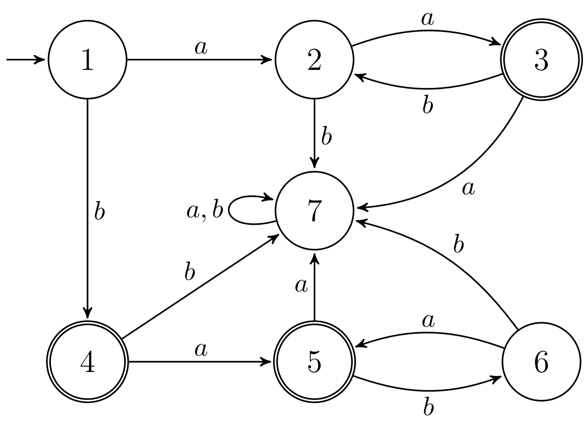
Exercise 5 : A rational half ?

Let L be a rational language over a finite alphabet Σ. Show that $\mathsf{Half}(L) = \{ f \in \Sigma^* : ff \in L \}$ is rational.

Is $\mathsf{FH}(L) = \{\ f \in \Sigma^* \ :\ \exists h \in \Sigma^*.\ |h| = |f|,\ fh \in L\ \}$ rational ?
Is $\mathsf{Double\text{-}pad}(L) = \{\ fh \in \Sigma^* \ :\ \ f \in L,\ h \in \Sigma^*,\ |h| = |f|\ \}$ rational ?

Exercise 6 : Minimization by Moore's algorithm

1. Minimize the automaton $\mathcal{A}_1$ using Moore's algorithm :



(a) Automaton $\mathcal{A}_1$

2. Give a minimal automaton for $\mathcal{L} = ((a(a+b)^2 + b)^* a(a+b))^*$.