Langages Formels - TD 3

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Exercise 1 : ANTIMIROV's automaton

The goal of this exercise is to build an automaton from a regular expression. We will define a *partial derivative* operation $\partial_a(E)$ which corresponds to $a^{-1}\mathcal{L}(E)$ (via interpretation of expressions). Formally, for every expression E and letter a, we define the *set of expressions* $\partial_a(E)$ as follows :

$$\begin{array}{rcl} \partial_a(\underline{\emptyset}) & = & \emptyset \\ \partial_a(\underline{b}) & = & \begin{cases} \emptyset & \text{if } a \neq b \\ \{\underline{\emptyset}^*\} & \text{else} \end{cases} \\ \partial_a(E+E') & = & \partial_a(E) \cup \partial_a(E') \\ \partial_a(E^*) & = & \partial_a(E) \cdot \{E^*\} \\ \partial_a(E \cdot E') & = & \begin{cases} \partial_a(E) \cdot \{E'\} & \text{if } \varepsilon \notin E \\ (\partial_a(E) \cdot \{E'\}) \cup \partial_a(E') & \text{else} \end{cases} \end{array}$$

where concatenation is naturally extended over sets of expressions.
We define $\partial_w(E)$ for a word w inductively with $\partial_\varepsilon(E) = \{E\}$ and $\partial_{wa}(E) = \partial_a(\partial_w(E))$, where $\partial_w(S) = \bigcup_{E \in S} \partial_w(E)$ when S is a set of expressions.
Given a set of regular expressions S, we denote by $\mathcal{L}(S)$ the set $\bigcup_{E \in S} \mathcal{L}(E)$.

1. Give the partial derivatives of $(ab+b)^*ba$ by a and b.
2. Prove that for every $L, L' \subseteq \Sigma^*$ and $a \in \Sigma$,

$$\begin{aligned} a^{-1}(L \cup L') &= (a^{-1}L) \cup (a^{-1}L') \\ a^{-1}L^* &= (a^{-1}L) \cdot L^* \\ a^{-1}(L \cdot L') &= \begin{cases} a^{-1}L \cdot L' & \text{si } \varepsilon \notin L \\ (a^{-1}L \cdot L') \cup (a^{-1}L') & \text{sinon} \end{cases} \end{aligned}$$

3. Show that $\mathcal{L}(\partial_w(E)) = w^{-1}\mathcal{L}(E)$.
4. We define the set of non empty suffixes of a word :

$$\mathrm{Suf}(w) = \{\ v \in \Sigma^+ \ :\ \exists u, w = uv\ \}$$

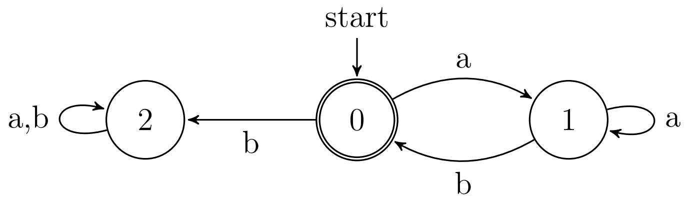
Show that for every $w \in \Sigma^+$:

$$\begin{array}{rcl} \partial_w(E+E') & = & \partial_w(E) \cup \partial_w(E') \\ \partial_w(E \cdot E') & \subseteq & (\partial_w(E) \cdot E') \cup \displaystyle\bigcup_{v \in \mathrm{Suf}(w)} \partial_v(E') \\ \partial_w(E^*) & \subseteq & \displaystyle\bigcup_{v \in \mathrm{Suf}(w)} \partial_v(E) \cdot E^* \end{array}$$

5. Let $\|E\|$ be the number of occurences of letters of Σ in E. Show that the set of partial derivatives different to E has at most $\|E\|+1$ elements.
6. Conclude, and apply the construction to the expression $(ab+b)^*ba$.

Exercise 2 : Transition monoid

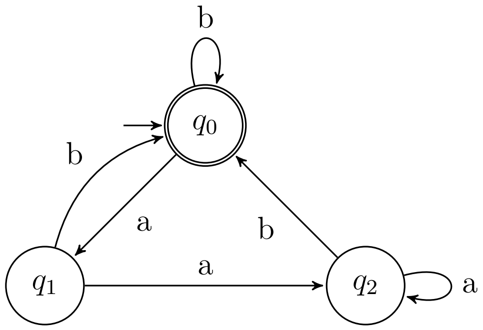
We consider the following finite deterministic complete automaton $\mathcal{A}$ over $\Sigma = \{a, b\}$:



1. Give $\mathcal{L}(\mathcal{A})$.
2. Give M the transition monoid of this automaton, a morphism ϕ and $P \subset M$ such that $L = \phi^{-1}(P)$.
3. What is the syntactical congruence of L ? What are its equivalence classes ?

Exercise 3 : Automate → Monoïde

Donnez le monoïde syntaxique M du langage $\mathcal{L}$ reconnu par cet automate, un morphisme ϕ et $P \subset M$ tel que $\phi^{-1}(P) = \mathcal{L}$.



Quelle est la congruence syntaxique de $\mathcal{L}$? Quelles sont ses classes d'équivalence ?

DM pour le 26 février 2026

Exercise 4 : Langages sans étoile

Soit $\Sigma = \{a, b\}$. La familles des *langages sans étoile* est la plus petite famille contenant $\emptyset$, $\{\ell\}$ pour tout $\ell \in \Sigma$, et close par union, complément et concaténation.

1. Exprimez les langages suivants comme langages sans étoile :
   (a) $\{\varepsilon\}$
   (b) les mots contenant bab
2. Exprimez $L_1 = (ab)^*$ comme un langage sans étoile.
3. Est-ce que $L_2 = (b^* + ab)^* + (b^* + ab)^* a$ est un langage sans étoile ?

Exercise 5 : State complexity of a language

Given a recognizable language L, we define its *state complexity* $\mathsf{Sc}(L)$ by the number of states of its minimal automaton. Show that the following inequalities hold (L^{t} is the transposed of L, the language of the mirror images of words of L) :

1. $\mathsf{Sc}(L \cap K) \leq \mathsf{Sc}(L)\mathsf{Sc}(K)$;
2. $\mathsf{Sc}(L \cup K) \leq \mathsf{Sc}(L)\mathsf{Sc}(K)$;

3. $\mathsf{Sc}(L^{\mathrm{t}}) \leq 2^{\mathsf{Sc}(L)}$;
4. $\mathsf{Sc}(LK) \leq (2\mathsf{Sc}(L) - 1)2^{\mathsf{Sc}(K)-1}$.

Not part of the exercise (i.e. you do not have to solve these questions, they are here to give you some food for thoughts), but interesting : we can show that some of these bounds have the right order of magnitude. Let $\Sigma = \{ a, b \}$.

5. Consider $L_n = \{|w|_a + |w|_b = 2n\}$ and $L'_n = \{|w|_a + 2|w|_b = 3n\}$ for the bound for intersection.
6. Consider $L_n = \Sigma^{n-1}a\Sigma^*$ for the bound for transposition.