Langages Formels
TD 4

Guillaume Scerri
(guillaume.scerri@lmf.cnrs.fr)

26 février 2026

Exercise 1 : Flashback

Vous avez montré que $Half(\mathcal{L}) = \{ u \in \Sigma^* \ : \ uu \in \mathcal{L} \}$ est régulier quand $\mathcal{L}$ est régulier. Si ce n'est pas déjà fait, montrer ce résultat en utilisant la reconnaissance par monoïdes.

Généraliser ce résultat à $\mathfrak{R}_k(L) = \{ u \ : \ u^k \in L \}$ en utilisant cette méthode.

Exercise 2 : Aperiodic languages

A language is *aperiodic* when its syntactic monoid $(M, \cdot)$ is aperiodic, i.e. for all $x \in M$ there exists $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

1. A finite deterministic complete automaton has a *counter* when there exists $n > 1$, a sequence of distinct states $q_0, \ldots, q_{n-1}$ and a word $w \in \Sigma^*$ such that $\delta(q_i, w) = q_{i+1 \mod n}$ for all $i \in \{ 0, \ldots, n-1 \}$.
   Show that $L \subseteq \Sigma^*$ is aperiodic iff its minimal automaton has no counter.
2. Show that if a language is *star-free*, i.e. in the smallest class containing the letters of the alphabet and closed by union, concatenation, and complement, then it is aperiodic.[1]
3. For the following languages, show if it is aperiodic or not :

   (a) $(ab)^*$,
   (b) $(aa)^*$,
   (c) $(a(ab)^*b)^*$,
   (d) $(ab + ba)^*$.

Exercise 3 : Selection property

A morphism $\mu : A^* \to B^*$ has the *selection property* iff for every regular language L, there exists a regular language $K \subseteq L$ such that μ is injective over K and $\mu(K) = \mu(L)$. The goal of this exercise is to show that every morphism has the selection property.

1. Show that all injective morphisms have the selection property.
2. Show that if morphisms μ and ν have the selection property, then the morphism $\mu \circ \nu$ also has it.

We call *projection* a morphism $\pi : A^* \to B^*$ such that for every letter $a \in A$, $\pi(a) = a$ or $\pi(a) = \varepsilon$.

3. Show that for every morphism $\mu : A^* \to B^*$, there exists an alphabet C, an injective morphism $\iota : A^* \to C^*$ and a projection $\pi : C^* \to B^*$ such that $\mu = \pi \circ \iota$.

We call *elementary projection* a projection $\pi : A^* \to B^*$ such that there exists a unique letter $a \in A$ such that $\pi(a) = \varepsilon$.

4. Show that every projection is the composition of elementary projections.

1. The converse also holds, but is much harder to prove.

5. Show that all elementary projection have the selection property.
6. Conclude.

Exercise 4 : Aperiodic languages

A language is *aperiodic* iff its syntactic monoid $(M, \cdot)$ is aperiodic, i.e. for all $x \in M$ there exists $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

1. A finite deterministic complete automaton has a *counter* if there exists $n > 1$, a sequence of distinct states $q_0, \dots, q_{n-1}$ and a word $w \in \Sigma^*$ such that $\delta(q_i, w) = q_{i+1 \mod n}$ for all $i \in \{ 0, \dots, n-1 \}$.
   Show that $L \subseteq \Sigma^*$ is aperiodic iff its minimal automaton has no counter.
2. For the following languages, show if it is aperiodic or not :
   (a) $(ab)^*$,
   (b) $(aa)^*$,
   (c) $(a(ab)^*b)^*$,
   (d) $(ab + ba)^*$.

Exercise 5 : Group language

A group language is a language whose syntactic monoid is a finite group. Show that a language is a group language if and only if it is recognised by a permutation automaton (i.e. for each letter a, the map $q \mapsto q \cdot a$ is a permutation of the set of states).

Exercise 6 : Characterizing recognizability

We want to show a converse to the pumping lemma. We say that a language L satisfies P_h if for all $uv_1 \dots v_h w$ avec $|v_i| \geq 1$, there exists $0 \leq j < k \leq h$ such that

$$uv_1 \dots v_h w \in L \Leftrightarrow uv_1 \dots v_j v_{k+1} \dots v_h w \in L.$$

The theorem of Ehrenfeucht, Parikh & Rozenberg states that L is rational iff there exists h such that L satisfies P_h.

1. Show that if L satisfies P_h, then $w^{-1}L$ also does for every word $w \in \Sigma^*$.
2. Let $h \in \mathbb{N}$. We want to show that the number of languages satisfying P_h is finite. We use the following statement of Ramsey's theorem :

   > For every k there is N such that, for every set E of cardinal greater than N and every bipartition $\mathcal{P}$ of $\mathfrak{P}_2(E) = \{ \{e, e'\} \ : \ e, e' \in E, \ e \neq e' \}$, there exists a subset $F \subseteq E$ of cardinal k such that $\mathfrak{P}_2(F)$ is contained in one of the classes of $\mathcal{P}$.

   Let N be the natural number given by Ramsey's theorem for $k = h + 1$. Let L and L' be two languages satisfying P_h and coinciding on words of size smaller than N. Prove that they coincide on words or size $M \geq N$, by induction on M. You may consider, for a word $f = a_1 \dots a_N t$ of size M (with $a_i \in \Sigma$), the following partition of $\mathfrak{P}_2([0; N])$:

   $$X_f = \{ (j, k) \ : \ 0 \leq j < k \leq N, \ a_1 \dots a_j a_{k+1} \dots a_N t \in L \}$$

   $$Y_f = \mathfrak{P}_2([0; N]) \setminus X_f$$

   Conclude.
3. Conclude that if a language L satisfies P_h for some h, then L is regular.

DM pour le 12 mars 2026

Exercise 7 : Congruences

Let's explore the fabulous world of congruences !

1. Using congruences, prove that $\{ xbx \ : \ x \in a^* \}$ is not regular.
2. Same question for any infinite subset of $\{ a^n b^n \ : \ n \in \mathbb{N} \}$.
3. Consider the regular language L represented by $a^*b^* + b^*a^*$.
   (a) Draw the minimal automaton for L.
   (b) Give a regular expression describing each of the equivalence classes of the syntactic congruence of L, denoted $\equiv_L$.
4. Let Σ be an alphabet. Let $\equiv$ be a a congruence of finite index over Σ^*. Prove that any equivalence class of $\equiv$ is a regular language of Σ^*.