Logic

Final Exam 2020

Results from the course can be used directly, but other results e.g. from the TD sessions need to be proved. Your answers need to be typeset using LaTeX and returned as a PDF file by email to `baelde@lsv.fr` before Tuesday, June 16th at 5pm. Questions about the exercises should be addressed to both `baelde@lsv.fr` and `dowek@lsv.fr`.

Proof transformations

Establish the following results about NK_1 derivations, for all Γ, ϕ, ϕ' and ψ. For the first two questions, you are not allowed to use the completeness of NK_1, so you will most likely have to reason over the structure of proofs. You cannot describe how each of the many rules is handled, but your proof should detail the important cases and explain why the others are unimportant or similar.

1. If $\phi \wedge \phi', \Gamma \vdash \psi$ is derivable then $\phi, \phi', \Gamma \vdash \psi$ is derivable.
2. If $\phi \vee \phi', \Gamma \vdash \psi$ is derivable then $\phi, \Gamma \vdash \psi$ and $\phi', \Gamma \vdash \psi$ are derivable.

Consider the following claims, give a counter-example when the claim is incorrect, otherwise justify why it is correct (possibly using the completeness of NK_1).

3. If $\phi \Rightarrow \phi', \Gamma \vdash \psi$ is derivable then $\phi', \Gamma \vdash \psi$ and $\Gamma \vdash \phi$ are derivable.
4. If $\exists x.\phi, \Gamma \vdash \psi$ is derivable then $\phi, \Gamma \vdash \psi$ is derivable.
5. If $\phi, \Gamma \vdash \psi$ is derivable then $\exists x.\phi, \Gamma \vdash \psi$ is derivable.

Flat Kripke structures

In this exercise we consider only propositional logic, i.e. formulas are built from propositional variables using boolean connectives only.

Let $\mathcal{K}$ be a Kripke structure with a set of worlds $\mathcal{W}$. We write $w < w'$ when $w \leq w'$ and $w \neq w'$. We say that $\mathcal{K}$ is *flat* when it does not contain three worlds w, w' and w'' such that $w < w' < w''$. Graphically, such structures look like this :

$$\begin{tikzcd} w_1 & w_2 & w_3 & \cdots \\ & w \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

We say that a sequent $\Gamma \vdash \phi$ is valid wrt. flat Kripke structures when, for any flat Kripke structure $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $\mathcal{K}, w \models \wedge\Gamma$, we also have $\mathcal{K}, w \models \phi$.

1. Show that the rules of propositional intuitionistic natural deduction (NJ_0) are sound wrt. flat Kripke structures : NJ_0 derivability implies validity wrt. flat Kripke structures.
2. Show that validity wrt. flat Kripke structures does not imply classical validity.
3. Show that NJ_0 is not complete wrt. flat Kripke structures.

Ehrenfeucht-Fraïssé games for two-variable logic

Consider the modification of Ehrenfeucht-Fraïssé games where Spoiler wins if, after some round $n \geq 2$ when $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1}), (a_n, b_n)$ have been played, the two pairs (a_{n-1}, b_{n-1}) and (a_n, b_n) do not induce a partial isomorphism. (In the standard games seen during the course, the partial isomorphism condition is only on the list of pairs from the last round. Here, we impose the condition for all rounds : it makes a difference because only the last two pairs are considered.)

Consider the following three structures over $\mathcal{F} = \emptyset$ and $\mathcal{P} = \{R\}$ where the interpretation of the binary symbol R is represented by arrows :

$$\mathcal{S}_2 \qquad a_1 \longrightarrow a_2$$

$$\mathcal{S}_3 \qquad b_1 \longrightarrow b_2 \longrightarrow b_3$$

$$\mathcal{S}_4 \qquad c_1 \longrightarrow c_2 \longrightarrow c_3 \longrightarrow c_4$$

1. Show that Duplicator does not have a winning strategy for two rounds over $\mathcal{S}_2$ and $\mathcal{S}_3$.
2. What is the largest number of rounds for which Duplicator has a winning strategy over $\mathcal{S}_3$ and $\mathcal{S}_4$?

$$\mathcal{Z} \qquad \cdots \longrightarrow c_{-1} \longrightarrow c_0 \longrightarrow c_1 \longrightarrow \cdots \longrightarrow c_n \longrightarrow c_{n+1} \longrightarrow \cdots$$

$$\mathcal{C}_n \qquad \begin{tikzcd} c'_1 \arrow[r] & c'_2 \arrow[r] & \cdots \arrow[r] & c'_{n-1} \arrow[r] & c'_n \arrow[llll, bend right] \end{tikzcd}$$

3. Let k be a natural number. What is the least n such that Duplicator has a winning strategy between $\mathcal{Z}$ and $\mathcal{C}_n$? Justify your answer.

The two-variable fragment of first-order logic is the set of formulas that use only two variables. For instance $R(x, y)$ and $R(x, y) \wedge \exists x.\ R(y, f(x))$ are two-variable formulas, but not $R(x, y) \wedge R(y, z)$ or even $R(x, y) \wedge \exists z.\ R(y, f(z))$ if x, y and z are distinct variables.

4. Give a two-variable formula ϕ that is closed and flat, and such that $\mathcal{S}_2 \models \phi$ and $\mathcal{S}_3 \not\models \phi$.
5. Give a two-variable formula ψ that is closed and flat, and such that $\mathcal{S}_4 \models \psi$ and $\mathcal{S}_3 \not\models \psi$.

Fix some finite $\mathcal{F}$ and $\mathcal{P}$. We write $\mathcal{S} \sim_k \mathcal{S}'$ when $\mathcal{S} \models \phi$ iff $\mathcal{S}' \models \phi$ for all closed flat two-variable formula ϕ of rank $\leq k$. It can be shown that, if Duplicator has a winning strategy on $\mathcal{S}$ and $\mathcal{S}'$ for n rounds, then $\mathcal{S} \sim_n \mathcal{S}'$: the argument is the same as for the analogue result for standard games.

6. The converse can also be proved by adapting the corresponding argument from the course : $\mathcal{S} \sim_n \mathcal{S}'$ implies that Duplicator has a winning strategy for n rounds. Explain how. You must precisely describe how the formulas $\phi_k^{b_1, \ldots, b_{n-k}}$ from the course can be adapted, and informally justify why the argument goes through with your modification. You do not need to *write* further details but be careful to *check* them, otherwise you might miss something in the adaptation of the formula !

Russell's paradox in Resolution

1. Let A be the comprehension axiom

$$\forall x\ (x \in R \Leftrightarrow \neg x \in x)$$

   In which language is the proposition A expressed ?

2. What is the clausal form of A ?
3. Provide a derivation in Resolution of the clause $\bot$, from this clausal form.
4. Show that there is no derivation using the Resolution rule only.
5. Let B be the separation axiom

$$\forall x\ (x \in R \Leftrightarrow (x \in E \wedge \neg x \in x))$$

   Express the proposition C : "the set E is not the set of all sets"

6. Find a Resolution proof of the sequent $B \vdash C$.

Consistency

Consider a language $\mathcal{L}$ that contains the symbols 0, S, and $\leq$ and a theory $\mathcal{T}$ in this language that has a model $\mathcal{M}$ such that

— the domain of the model is the set $\mathbb{N}$,
— the interpretation of the symbol 0 is the natural number 0,
— the interpretation of the symbol S is the function $n \mapsto n + 1$,
— the interpretation of the symbol $\leq$ is the usual order on natural numbers.

Consider the language $\mathcal{L}' = \mathcal{L} \cup \{c\}$ and the theory $\mathcal{U}$ containing the an infinity of axioms of axioms

$$0 \leq c$$

$$S(0) \leq c$$

$$S(S(0)) \leq c$$

$$S(S(S(0))) \leq c$$

$$\ldots$$

1. Build a model of the theory $\mathcal{T} \cup \{S(0) \leq c, S(S(S(S(0)))) \leq c\}$.
2. Let $\mathcal{F}$ be any finite subset of $\mathcal{U}$. Build a model of the theory $\mathcal{T} \cup \mathcal{F}$.
3. Is there a proof of the proposition $\bot$ in the theory $\mathcal{T} \cup \mathcal{F}$?
4. Is there a proof of the proposition $\bot$ in the theory $\mathcal{T} \cup \mathcal{U}$?
5. Does the theory $\mathcal{T} \cup \mathcal{U}$ have a model ?
6. Does the theory $\mathcal{T} \cup \mathcal{U}$ have a model where the domain is $\mathbb{N}$, the interpretation of the symbol 0 is the natural number 0, the interpretation of the symbol S is the function $n \mapsto n + 1$, and the interpretation of the symbol $\leq$ is the usual order on natural numbers ?