Measuring well quasi-orders and complexity of verification

Mesure des beaux pré-ordres et complexité de la vérification

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THESE DE DOCTORAT

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Titre: Mesure des Beaux Pré-Ordres et complexité de la vérification

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Nous étudions trois mesures ordinales des beaux pré-ordres, aussi appelées invariants ordinaux: le type d'ordre maximal, la largeur et la hauteur. Un enjeu principal est de calculer les invariants ordinaux de beaux pré-ordres complexes construits à partir de beaux pré-ordres plus simples grâce à des opérations classiques, tel que le produit Cartésien, ou encore des opérations d'ordre supérieur comme le plongement sous-mot sur les mots finis.

Dans cette thèse, nous calculons compositionellement le type d'ordre maximal du produit direct, la largeur de l'ordre plongement sur les multisets, et les largeur et hauteur de l'ordre Dershowitz-Manna sur les multisets finis. De plus, nous calculons la largeur du produit Cartésien dans des cas restreints, et prouvons des bornes atteintes pour les mesures ordinales du powerset fini.

Pour ce faire, nous développons plusieurs outils et techniques, notamment une approche par jeu et stratégies pour calculer la largeur, amenant à la notion de famille de sous-ensembles quasi-incomparables. Pour la largeur de l'ordre Dershowitz–Manna, nous introduisons et étudions un quatrième invariant ordinal, le type d'ordre amical.

Pour illustrer l'intérêt de nos résultats, nous délimitons une grande famille de beaux pré-ordres élémentaires, construits à partir d'ordinaux et de constructions classiques, dont les invariants sont connus.

Title: Measuring well quasi-orders and complexity of verification

Keywords: well quasi-orders, ordinal measures, maximal order type, ordinal width, ordinal height

Abstract: We investigate three ordinal measures of a well quasi-order, also called ordinal invariants: maximal order type, width, and height. One main challenge is to compute the ordinal invariants of complex well quasi-orders built from simpler well quasi-orders through classical operation, such as the Cartesian product, and high-order constructions, like the finite words embedding.

In this thesis, we compute compositionally the maximal order type of the direct product, the width of the multiset embedding, and the height and width of the multiset ordering. Furthermore, we compute the width of the Cartesian product in restricted cases and prove tight bounds on the ordinal measures of the finite powerset.

In the process, we develop several tools and techniques, notably a game-theoretical approach to computing width using the notion of quasi-incomparable families of subsets. To tackle the width of the multiset ordering, we introduce and study a fourth ordinal invariant, the friendly order type.

As an illustration of the power of our results, we delimit a large family of elementary wqos, built from ordinals and classical constructions, whose ordinal invariants are known.

Contents

Introduction

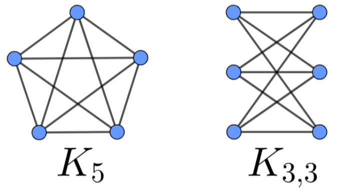
Well quasi-orders

Well quasi-orders (wqos) are quasi-orders that are well-founded (no infinite decreasing sequences) and verify the finite antichain condition (no infinite set of pairwise incomparable elements). Wqos can equivalently be characterized as quasi-orders that have no infinite bad sequences, where a sequence (x_i) is bad if for all $i < j$, $x_i \not\leq x_j$. Well quasi-orders are an intermediate notion between well orders and well-founded quasi-orders: Any well-order is a well quasi-order, and any well quasi-order is well-founded. Intuitively, wqos can be seen as finitary constraints on infinite sets.

The notion of well quasi-orderedness was introduced by Higman (1952); Rado (1954) under the name "finite basis property". Some previous articles had hinted at this notion, for instance Dickson (1913) which proved that fixed-dimension tuples of integers ordered component-wise are wqos as a useful lemma. Wqos were then rediscovered several times (Kruskal, 1972), for instance by Haines (1969) who proved a weaker version of Higman's lemma, applied to formal languages. Well quasi-orders appear in many areas of mathematics and computer science: combinatorics, logic, topology, automata theory, proof theory, term rewriting, graph theory, program verification, and more, where they help tackle finiteness problems, decidability questions, and proof/program termination. Kříž and Thomas (1990a) cites four reasons that one might be interested in wqos:

1. It is fun.

2. One can deduce "excluded minor theorems". For instance, the Robertson-Seymour Theorem, which states that the minor ordering on finite graph is a wqo, implies that any set of finite graphs closed by minors can be characterized by a finite set of excluded minors, just like the following characterization of planar graphs: planar graphs are exactly graphs that do not contain K_5 and $K_{3,3}$ as minors (Kuratowski, 1930; Wagner, 1937).



3. Well quasi-orderedness can be used to prove complexity upper bounds in a non-constructive way. This point is derived from the previous point: if some set of elements is characterized by a finite number of excluded minors, and there is a polynomial procedure to decide if some element (given as input) has for minor some other (fixed) element, then the membership problem is decidable with a polynomial-time algorithm (and you do not need to know the excluded minors of your set, nor build the algorithm). For instance, the membership problem for a language closed by subwords is decidable in LOGSPACE since one can test if a given word accept a fixed word as subword in LOGSPACE.

4. From the viewpoint of reverse mathematics, the theoretic strength of results on wqos (i.e., which axioms of second-order arithmetic are needed to prove them, and conversely which axioms can we derive from these results) span across the Big Five[1]. For instance, Higman's Lemma is equivalent to ACA_0, Kruskal's Tree Theorem implies ATR_0 whereas Robertson-Seymour graph minor theorem is not provable with $\Pi^1_1{-}\mathrm{CA}_0$, the strongest fragment of the Big Five (see Marcone (2020) for a survey of reverse mathematics on wqos).

I wholeheartedly agree with the first reason. Furthermore, more applications for wqos arise when one is given tools to measure them.

Measuring wqos

There are intuitive notions of measures on a finite qo: its cardinal, its height (the size of the largest decreasing sequence), and its width (the size of the largest antichain). Atkinson (1989) introduced two more measures: the number of relations of a finite qo, and its number of possible linearisations.

The notions of cardinal, height, and width can be generalized to infinite qos, under certain conditions. In a well-founded qo, the ordinal height is the supremum of the order types of its chains. Wolk (1967) showed that in a wqo, this height is reached by a maximal chain. Similarly, de Jongh and Parikh

[1] The Big Five is the name given to five fragments of second-order arithmetic; most results in ordinary mathematics are equivalent to one of these fragments.

(1977) introduced the maximal order type (mot) as the supremum of the order types of the linearisations of a wqo, and proved that the mot is reached by a maximal linearisation. Kříž and Thomas (1990b) introduced equivalent characterizations for mot and height as ranks of trees of bad sequences and decreasing sequences, respectively, which naturally led to the definition of a third ordinal invariant, width, the rank of the tree of antichains. Less studied than its counterparts, the width of a wqo generalizes the width of a finite qo, but contrary to heigth and mot, cannot be defined as the supremum of the order type of its antichains. While exploring techniques for program termination, Blass and Gurevich (2008) developed these characterizations into a game-theoretical point of view which gives useful intuitions on how to compare and measure wqos.

When going from finite to infinite qos, the number of relations is not a notion that makes much sense for infinite qos (it would be a cardinal). In the spirit of the number of possible linearisations of a finite qo, Dushnik and Miller (1941) studied the dimension of a partial order, a cardinal which represents the minimal number of linearisations needed to "realize" a partial order, i.e. such that their intersection gives the partial order. Similarly, the covering number of a partial order is a cardinal invariant which is the least number of chains needed to cover a partial order. Abraham and Pouzet (2023) studied the covering number in the case of partial order with the finite antichain condition. In this thesis we chose to focus on ordinal invariants only.

Application to computer science

Program termination

One way to prove that a program does not terminate is to find a configuration c_1 reachable from the initial configuration of the program c_0, such that from configuration c_1 the program can loop back to c_1. This is not the only way: for instance, if from c_1 the program can reach a configuration c_1' that simulates c_1 (i.e.from c_1' the program can imitate the path of the program from c_1), then the program does not terminate.

Now consider a program where the simulation relation between configurations is wqo. Then either the path from c_0 is finite, thus the program terminates, or this path is infinite hence there exists $c_1 \to^* c_1'$ such that $c_1 \leq c_1'$ for the simulation order. Hence the method to determine the termination of this program is complete.

This is the motivating idea behind the formalism of well-structured transition systems.

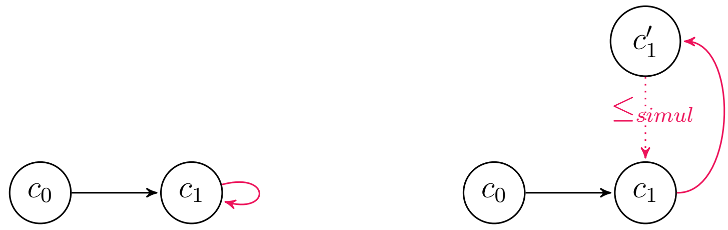


Figure 1: Proof of non-termination from loop and loop simulation.

Well-structured transition systems

Well-structured transition systems (wsts) are infinite-state transition systems whose set of configurations can be ordered with a wqo $\leq$ such that $\leq$ is a (weak) simulation relation. For instance, vector addition systems with states (VASS) have $\mathbb{N}^d \times Q$ as their set of configurations (where Q is the set of control states, d is the number of counters), where the component-wise order is a weak simulation. The well quasi-orderedness of the configuration set can then be used to prove decidability for some classical verification problems, such as termination, safety, simulation, etc. Finkel (1987) gave a first definition of wsts, studying Petri nets, then used this notion in Finkel (1994) for lossy channel systems. Independently, Abdulla and Jonsson (1996a,b) invented the backward-chaining algorithm, to study coverability of lossy channel systems. Abdulla et al. (2000) and Finkel and Schnoebelen (2001) then generalized and popularized wsts, giving a flurry of examples: VASS, priority channel systems, timed automata, context-free grammars, ...

Complexity questions follow naturally from decidability, and can be answered using ordinal invariants. Bonnet et al. (2013) observed that simulation relations between wsts is equivalent to reflections between their underlying wqos, which means that one can compare the expressiveness of two wsts by comparing maximal order types. However, we can leverage ordinal measures one step further, using ordinal-indexed complexity classes.

A well-known technique to prove hardness result is to show that a model can weakly compute some class of recursive functions. For instance, Mayr and Meyer (1981) built Petri nets that weakly compute Ackermann-related functions to prove that the finite containment problem for Petri nets is Ackermann hard. Similarly, Schnoebelen (2002) built lossy channel systems that compute Ackermann-related functions and their inverse (necessary to deal with the "lossy" feature of lossy channel systems), reinventing a result from Urquhart (1999) from the domain of relevance logic. Then Chambart and Schnoebelen (2008) built lossy channel systems that computed hyper-Ackermannian functions and their inverse. This technique also appears in Leroux et al. (2019) for grammar-controlled VASS.

Function hierarchies can also be used to prove complexity upper bounds.

McAloon (1984); Clote (1986) used the Grzegorczyk hierarchy to give an upper bound to Mayr and Meyer (1981)'s result. Chambart and Schnoebelen (2008) gave an hyper-Ackermannian upper bound for Lossy Channel Sytems using the fast-growing hierarchy—a hierarchy of complexity classes $(\mathcal{F}_\alpha)_\alpha$ indexed by ordinals based on the Wainer-Cichon hierarchy of recursive functions—through the technology of controlled bad sequences.

A controlled sequence on a normed wqo is a sequence whose elements have their norm bounded by a control function, which means their length is bounded. In practical examples, the length of controlled bad sequences is bounded through some length function theorem by the mot of the underlying wqo[2], with a statement of the form: If the mot of a wqo is ω^β then the length of controlled bad sequences of this wqo is a function of F_β (to give landmarks, $\mathcal{F}_3$ = Tower, $\mathcal{F}_\omega$ = Ack and $\mathcal{F}_{\omega^\omega}$ = HAck). Thus the maximal order type can be seen as a "rule of thumb" measure to assess the complexity of a wsts, though the details of a specific length function theorem have to be tailored to the wsts studied. The term "Length Function Theorem" is coined first in Schmitz and Schnoebelen (2011), who give a Length Function Theorem for controlled bad sequences on an elementary family of normed wqos. This theorem is used in Haddad et al. (2012) to prove an upper bound on the complexity of enriched nets. Figueira et al. (2011); Haase et al. (2014); Balasubramanian (2020) also developed length function theorems, for tuples of integers and priority channel systems. See Schmitz (2016) to for a formalisation of complexity hierarchies and length function theorems.

In some cases, one can give upper bounds on controlled bad sequences by factorizing them in controlled antichains (Schmitz, 2019) or studying decreasing sequences (Abriola et al., 2015). Therefore all three ordinal invariants are relevant to complexity analysis.

Reverse mathematics

Ordinal invariants also give new insight into the proof-theoretic strength of results on wqos. In a given fragment of ordinal arithmetic, a theorem is equivalent to the assertion that some ordinal is well-founded when you can prove either one from the other. For instance, Simpson (1988) proved that the Hilbert finite basis theorem is equivalent to the well-foundedness of ω^ω. Moreover, Friedman (Simpson, 1985) and Rathjen and Weiermann (1993) showed that, within ACA_0, the well-quasi-orderedness of some specific embedding orderings is equivalent to the well-foundedness of their maximal order types.

[2]Unfortunately, one cannot expect a general theorem linking complexity to mot, as the length of controlled bad sequences depends not only on the underlying wqo, but also its norm and the control function, which have to fit the wsts studied.

Measuring wqos compositionally: state of the art

A natural way to study wqos invariants is to measure the ordinal invariants of various well-ordered data structures: de Jongh and Parikh (1977) computed the mot of the disjoint sum and the Cartesian product of wpos, and the mot of the finite word embedding on a finite alphabet. Schmidt (1979) then computed the mot of the word embedding in the general case and the homeomorphic tree embedding on a wpo. Abraham and Bonnet (1999) pursued this line of study by computing the height of the Cartesian product, but also the width of the disjoint sum and lexicographic product. Weiermann (2009) built on Aschenbrenner and Pong (2004) to compute the maximal order type of the multiset embedding. Van der Meeren et al. (2015) completes Weiermann (1991) to obtain the m.o.t. of the multiset ordering. For a complete survey of these results see Džamonja et al. (2020), who computed the ordinal invariants of the lexicographic product, but also the height of the multiset word and tree embeddings. Džamonja et al. (2020) also showed that many gaps remain. In this thesis, we pursue this line of study by computing the ordinal invariants of several wqos constructions.

Contributions

Chapter 1 recalls well-known definitions and results regarding wqos. In Chapter 2 are gathered all the tools used in this thesis, some known techniques as well as original results.

We then leverage these tools to study the maximal order type of the direct product (Chapter 3), the width of the Cartesian product (Chapter 4), the width of the multiset embedding and the width and height of the multiset embedding (Chapter 5), and the ordinal invariants of the finite powerset (Chapter 6). In Chapter 7, we build a family of elementary wqos, inspired from the family studied in Figueira et al. (2011), closed under usual constructors on wqos. We then test the limit of this elementary family in Chapter 8.

How to read this dissertation

Chapters dependancies. Chapters 1 and 2 give definitions, notations and tools used throughout this dissertation. Chapters 3 to 6 can be read independently. Chapter 7 relies on results proven in Chapters 4 to 6. Chapter 8 only depends on Chapter 4.

Framed results. The results we believe are the most important are framed so they can be easily spotted. Some new results are not framed, often because they are stepping stones towards more interesting results.

Ordinal arithmetic. To compute ordinal invariants, one need to be fluent in ordinal arithmetic. Since operations on ordinals can be quite tricky, we refer to Appendix A for any questions.

Correspondence between chapters and articles. Three papers (Vialard, 2024, 2023; Abriola et al., 2023) have been published in the scope of this PhD. They correspond roughly to Chapter 4, Chapter 5, and Chapters 6 and 7, respectively. Chapter 2 contains tools developed across all three articles. However, this dissertation contains several results that are not in any of these articles (but might be the subject of future articles).

Notation conventions. In this thesis, Greek letters refer to ordinals, capital letters A, B, C, X to well quasi-orders, lowercase letters k, l, m, n to integers, a, b, c, x, y to elements of a wqo.

1 - Basics

1.1 . Well quasi-orders

A quasi-order (or preorder) is a reflexive and transitive binary relation $\leq_A$ on a set A. A partial order is a quasi-order with antisymmetry. We say that $(A, \leq_A$ is a quasi-ordering (resp. a partial ordering). We often write only A when the underlying order is understood.

When any two elements in a quasi-ordered or partially ordered set are comparable, then it is a linear (or total) (quasi-)ordering.

A strict order can be defined from $\leq_A$: $x <_A y$ means that $x \leq_A y$ and $x \not\geq_A y$. We write the incomparable relation $x \perp_A y$ when $x \not\leq_A y$ and $y \not\leq_A x$. Similarly, the equivalence relation $x \equiv_A y$ means that $x \leq_A y$ and $y \leq_A x$.

A sequence $x_1, \ldots, x_n$ in a qo A is an antichain sequence if the set $\{ x_1, \ldots, x_n \}$ is an antichain, i.e. for all $i < j$, $x_i \perp_A x_j$, a decreasing sequence if for all $i < j$, $x_i >_A x_j$, a bad sequence if for all $i < j$, $x_i \not\geq_A x_j$.

We have to choose whether to develop a theory around partial orders or quasi-orders. Both have their advantages. Partial orders are more intuitive, easier to grasp. However some classical operations on partially ordered sets produce quasi-ordered sets, like the finite powerset construction, that will play a major part in this manuscript.

Example 1.1.1. *For any quasi-ordering $(A, \leq_A)$, let $(\mathsf{P_f}(A), \sqsubseteq_{\mathcal{H}})$ be the set of* finite *subsets of A ordered with* Hoare's embedding*:*

$$S \sqsubseteq_{\mathcal{H}} S' \iff \forall x \in S, \exists y \in S', x \leq_A y \ .$$

$(\mathbb{N}, \leq)$ is a linear ordering but $\mathsf{P_f}(\mathbb{N})$ is a linear quasi-order: $\{ 3 \} \equiv \{ 0, 3 \} \equiv \{ 1, 3 \} \equiv \{ 0, 2, 3 \} \equiv \ldots$

Fortunately, a qo A naturally gives rise to a partial ordering when we consider its quotient by the underlying equivalence $\equiv_A$. This quotienting preserves the ordinal invariants we will introduce next.

Example 1.1.2. *All elements of an equivalence class of $\mathsf{P_f}(A)$ have the same maximal elements, which form an antichain. Hence the quotient $(\mathsf{P_f}(A), \sqsubseteq_{\mathcal{H}})_{/\equiv_{\mathcal{H}}}$ is isomorphic to $(\mathsf{Ant}(A), \sqsubseteq_{\mathcal{H}})$ the set of finite antichains of A ordered with Hoare's embedding.*

In particular, $(\mathsf{P_f}(\mathbb{N}), \sqsubseteq_{\mathcal{H}})_{/\equiv_{\mathcal{H}}}$ is isomorphic to $(\mathsf{Ant}(\mathbb{N}), \sqsubseteq_{\mathcal{H}})$ which is trivially isomorphic to $(\mathbb{N}, \leq)$.

We say that two quasi-orders are quasi-isomorphic when the partial orderings obtained through quotienting by the equivalence relation are isomorphic. We denote isomorphicity and quasi-isomorphicity with $\cong$, indiscriminately.

A linearisation of a quasi-order $(A, \leq)$ is a linear quasi-order $(A, \preceq)$ with same carrier set such that for any elements $x, y \in A$, $x \leq y \implies x \preceq y$.

Characterization of wqos. The following characterisations are equivalent (Fraïssé, 1986):

1. A is a well quasi-ordering.
2. A is well-founded and verifies the finite antichain condition (FAC), i.e. has neither infinite decrasing sequences nor infinite antichains.
3. A has no infinite bad sequences.
4. A has the finite basis property, i.e., for any non-empty subset of A, the set of minimal elements is finite and non-empty.
5. Every infinite sequence of A contains an increasing subsequence.

A wqo with antisymmetry is a well partial ordering (wpo).

Remark 1.1.3. The concept of wqo has been rediscovered many times, hence it had many names: Higman (1952) uses the term "quasi-order with finite basis property", as another equivalent characterisation of wqo is a qo where every subset has at least one and only a finite number of minimal elements. Other contenders for the name were "partially well-ordered" (Rado, 1954), "tight partial order", "fairly well-ordered"... The name wqo seems to come from Kruskal (1960).

This multiplicity of names hints at the importance and intuitive nature of the concept. The obvious drawback is that a reader often need a glossary from one article to another (like the glossary at the end of Kruskal (1960), which maps his terminology with the one in Higman (1952)).

Other notions used here have had several names: for instance, a qo with FAC can be said to be "finitely free", bad sequences are sometimes called "non-dominating", etc. We will try to give these connections when we know them.

1.2 . Ordinal invariants

Rank of well-founded trees. To define the ordinal invariants of wqos, we must first introduce the notion of rank on well-founded rooted trees.

One can see a tree as a partial ordering where the relation "x is a child of x'" induces the order $x \leq x'$. Thus a well-founded rooted tree is a tree that has no infinite branches. Classically, one ascribes an ordinal rank r to any node of a well-founded tree T from leaves to root. Let $s \in T$ be a node: if s is a leaf, then $r(s) \stackrel{\text{def}}{=} 0$, otherwise $r(s) \stackrel{\text{def}}{=} \sup\{r(t) + 1 \mid t \text{ is a child of } s\}$. Since T can be infinitely branching, $r(s)$ is an ordinal. The rank of T is defined as the rank of its root.

Lemma 1.2.1 (Wolk (1967))**.** *Let T be a tree of rank α, and $\beta < \alpha$. Then there is a node $s \in T$ such that $r(s) = \beta$.*

Proof. We prove that, for all $s \in T$, by induction on $r(s)$ that for all $\beta < r(s)$ there is a node $t \leq_T s$ such that $r(t) = \beta$. Obviously it is true for $r(s) = 0$.

If $r(s) = \gamma + 1$ then there exists some t a child of s such that $r(t) = \gamma$ and for any $\beta < \gamma$ there exists $u \leq t \leq s$ such that $r(u) = \beta$ by induction hypothesis.

If $r(s) = \lambda$ a limit ordinal and $\beta < \lambda$ then there exists some child t of s such that $\beta \leq r(t) < \lambda$. Hence by induction hypothesis there exists $u \leq t \leq s$ such that $r(u) = \beta$. □

Ordinal invariants as rank of trees. For any wqo A, $\mathsf{Inco}(A)$ (resp. $\mathsf{Dec}(A)$ and $\mathsf{Bad}(A)$) is the tree of antichain sequences (resp. strictly decreasing sequences, bad sequences) of A ordered by inverse prefix order ($s \leq s'$ iff s' is a prefix of s). Here the empty sequence is the root, and if s is the maximal strict prefix of t, then t is a child of s. These trees were first introduced by Kříž and Thomas (1990b) along with other trees of sequences on wqos.

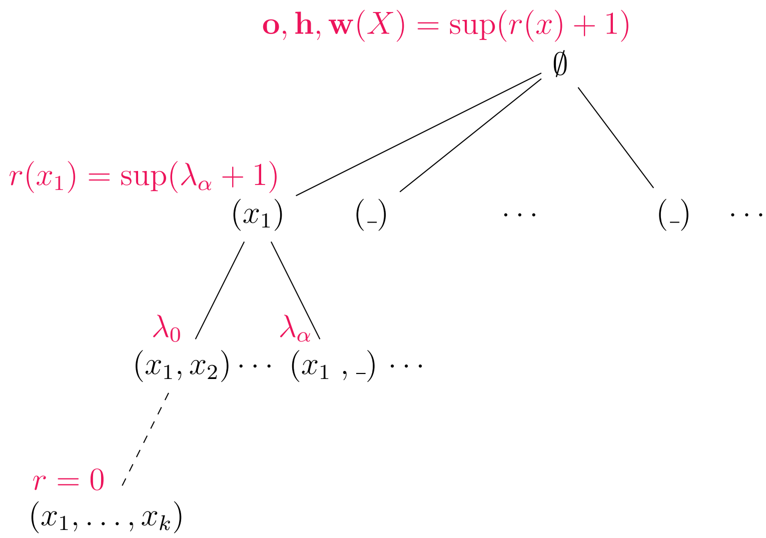


Figure 1.1: Ascribing rank of to a tree of sequences from bottom to top.

Observe that, since A is a wqo (hence FAC and WF), the trees $\mathsf{Inco}(A)$,

$\mathsf{Dec}(A)$ and $\mathsf{Bad}(A)$ do not have infinite branches: they are well-founded. However, they can be infinitely branching, thus their rank can be infinite.

Definition 1.2.2 (Ordinal invariants, Kříž and Thomas (1990b))**.** The *width* $\mathbf{w}(A)$, the *height* $\mathbf{h}(A)$, and the *maximal order type* $\mathbf{o}(A)$ of a wqo A are the ranks of $\mathsf{Inco}(A)$, $\mathsf{Dec}(A)$, and $\mathsf{Bad}(A)$, respectively. Together, they are called the *ordinal invariants* of A.

Example 1.2.3. *We identify each ordinal as the set of its predecessors ordered with inclusion. Therefore an ordinal* $\alpha > 0$ *is a* wqo *of ordinal invariants* $\alpha > 0$, $\mathbf{o}(\alpha) = \mathbf{h}(\alpha) = \alpha$, *and* $\mathbf{w}(\alpha) = 1$.

Example 1.2.4. *For any* $0 < n < \omega$, *let* Γ_n *denote a set of* n *incomparable elements. Then* $\mathbf{o}(\Gamma_n) = \mathbf{w}(\Gamma_n) = n$, *and* $\mathbf{h}(\Gamma_n) = 1$.

Since antichain sequences and strictly decreasing sequences are bad sequences, $\mathsf{Inco}(A)$ and $\mathsf{Dec}(A)$ are subtrees of $\mathsf{Bad}(A)$. Hence:

Lemma 1.2.5. *For all wqo* A, $\mathbf{w}(A) \leq \mathbf{o}(A)$ *and* $\mathbf{h}(A) \leq \mathbf{o}(A)$.

Definition 1.2.2 has the advantage of giving homogeneous characterization for all three ordinal invariants. However, mot and height add historically other characterizations, that Kříž and Thomas (1990b) proved to be equivalent.

Theorem 1.2.6 (Minimax theorem for mot)**.** *Let* A *be a* wqo *and let* $\alpha = \sup\{\, \gamma \mid \gamma$ *is a linearisation of* $A\,\}$.

- *There exists a maximal linearisation* A' *of* A *such that* $A' \cong \alpha$ *(de Jongh and Parikh, 1977),*
- *and* $\mathbf{o}(A) = \alpha$ *(Kříž and Thomas, 1990b).*

Theorem 1.2.7 (Minimax theorem for height)**.** *Let* A *be a* wqo *and let* $\beta = \sup\{\, \gamma \mid \gamma$ *is a chain of* $A\,\}$.

- *There exists a maximal chain* A' *of* A *such that* $A' \cong \beta$ *(Wolk, 1967),*
- *and* $\mathbf{h}(A) = \beta$ *(Kříž and Thomas, 1990b).*

Since antichains are set of unordered elements, there can be no notion of order type of an antichain. Therefore there is no minimax theorem for width. This is why we favour Definition 1.2.2, which gives with an homogeneous definition of all three invariants. Moreover, it allows defining more invariants, as there are many well-founded trees of class of sequences on a wqo (see for instance the friendly order type in Chapter 5.

Remark 1.2.8. Schmidt (1979) wondered about the significance of a *minimal* order type, the minimum of the order types of the linearisations of a well partial order A. She proved that it is bounded between $\mathbf{h}(A)$ and $\mathbf{h}(A) + \omega$. Since it quite similar to the height, we choose to ignore it.

Residual characterization of ordinal invariants. For a quasi-order A, an alement $x \in A$, and a relation symbol $* \in \{\perp, <, >, \not\leq, \not\geq\}$, we define the $*$-residual of A at x as

$$A_{*x} = \{y \in A : y * x\} \ .$$

We can generalize this notion to subsets $Y \subseteq A$:

$$A_{*Y} = \bigcap_{x \in Y} A_{*x} \ .$$

If $Y = \emptyset$, $A_{*Y} = A$.

For instance, $\mathbb{N}_{<2} = \{0, 1\}$ and $\mathbb{N}_{\perp 2} = \emptyset$. In Figure 1.2, you can see the residuals of $(\mathbb{N}^2, \leq_\times)$ (i.e. pairs of integers ordered component-wise), at $x = (4, 6)$: $\mathbb{N}^2_{<x}$ in empty blue circles, $\mathbb{N}^2_{>x}$ in green circles with stars, and $\mathbb{N}^2_{\perp x}$ in filled red circles. The union of the red and blue parts is $\mathbb{N}^2_{\not\geq x}$, the union of the red and green parts is $\mathbb{N}^2_{\not\leq x}$.

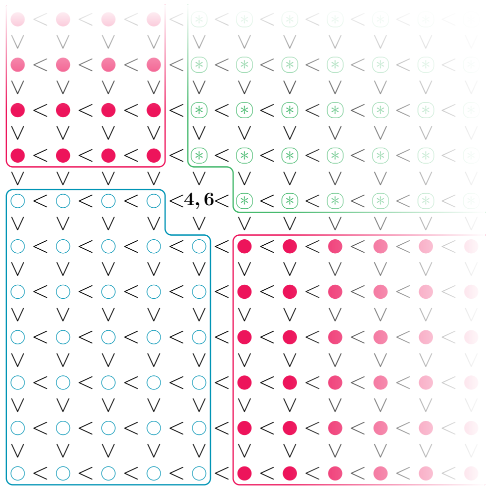


Figure 1.2: Residuals of $(\mathbb{N}^2, \leq_\times)$ at $(4, 6)$.

From here on, when we consider a subset A' of a qo $(A, \leq_A)$, we implicitly mean that A' is ordered with $\leq_A$ restricted to A'. Thus A' is a wqo when A is, with its own ordinal invariants, smaller than or equal to the ordinal invariants of A (see Lemma 2.1.5). This applies to residuals of A, since they are subsets of A.

Residuals are essential in the computation of invariants, given the following descent equations:

$$\mathbf{o}(A) = \sup_{x \in A} \{\mathbf{o}(A_{\not\geq x}) + 1\} \qquad \text{(Res-o)}$$

$$\mathbf{h}(A) = \sup_{x \in A} \{\mathbf{h}(A_{<x}) + 1\} \qquad \text{(Res-h)}$$

$$\mathbf{w}(A) = \sup_{x \in A} \{\mathbf{w}(A_{\perp x}) + 1\} \qquad \text{(Res-w)}$$

Remark 1.2.9. The notion of residual appear with different names in many articles. For instance, in de Jongh and Parikh (1977), $A_{\not\geq x}$ and $A_{\geq x}$ are denoted with $L_A(x)$ and $U_A(x)$, respectively. In de Jongh and Parikh (1977), where $\mathbf{o}(A)$ is defined as the supremum of the order types of linearisations of A, Equation (Res-o) is an essential theorem since it allows to prove that there is a maximal linearization of A whose order type reaches the $\mathbf{o}(A)$.

These formulas can be seen as a reformulation of tree rank computation: For any bad sequence s (resp. decreasing sequences, antichains), $r(s) = \mathbf{o}(A_{\not\geq s})$ (resp. $\mathbf{h}(A_{<s})$, $\mathbf{w}(A_{\perp s})$) in $\mathsf{Bad}(A)$ (resp. $\mathsf{Dec}(A)$, $\mathsf{Inco}(A)$) (see Section 2.3. of Džamonja et al. (2020)). Thus we can reformulate Lemma 1.2.1 using residuals:

Lemma 1.2.10. *For any ordinals $\beta \leq \alpha$, any $\mathbf{f} \in \{ \mathbf{o}, \mathbf{w}, \mathbf{h} \}$, and any wqo A such that $\mathbf{f}(A) = \alpha$, there exists a finite $Y \subseteq A$ such that $\mathbf{f}(A_{*Y}) = \beta$, with $* \in \{ \not\geq, \perp, < \}$ corresponding to f. Moreover, if $\mathbf{f} = \mathbf{h}$ there exists $x \in A$ such that $\mathbf{h}(A_{<x}) = \alpha$.*

The height behaves differently because the residual "strictly below" of a decreasing sequence is uniquely determined by the last element of this sequence.

We can use the descent equations Res-o, Res-h and Res-w to recursively compute the invariants of A: this is called the method of residuals.

Example 1.2.11. *For all $x = (n, m) \in \mathbb{N}^2$, $\mathbb{N}^2_{<x}$ is finite and contains decreasing sequences of maximal length $n + m$ (see Figure 1.2). Therefore $\mathbf{h}(\mathbb{N}^2_{<x}) = n + m$, hence $\mathbf{h}(\mathbb{N}^2) = \sup_{n,m\in\mathbb{N}} n + m = \omega$.*

Similarly, $\mathbb{N}^2_{\perp x}$ has antichains of at most $n + m$ elements (take the anti-diagonal), hence a width of $n + m$, therefore $\mathbf{w}(\mathbb{N}^2) = \sup_{n,m\in\mathbb{N}} n + m = \omega$.

For the the mot*, observe that $\mathbb{N}^2_{\not\geq x}$ contains n columns and m rows, all isomorphic to ω. Therefore its* mot *is $\omega \cdot (n + m)$. Thus $\mathbf{o}(\mathbb{N}^2) = \sup_{n,m\in\mathbb{N}} \omega \cdot (n + m) = \omega^2$.*

Remark 1.2.12. The downward-closure $\downarrow S$ and upward-closure $\uparrow S$ can be seen as some cousins of residuals for S a subset of a wqo A. $\downarrow S$ (resp. $\uparrow S$) means the smallest downward-closed (resp. upward closed) subset of A that contains S. Hence $\downarrow S \stackrel{\mathsf{def}}{=} \bigcup_{x\in S} A_{\leq x}$ and $\uparrow S \stackrel{\mathsf{def}}{=} \bigcup_{x\in S} A_{\geq x}$ are a union of residuals, whereas $A_{*S} = f \bigcap_{x\in S} A_{*x}$ is an intersection of residuals. Therefore $A \setminus \uparrow S = A_{\not\geq S}$ and $A \setminus \downarrow S = A_{\not\leq S}$. Notice that the downward (resp. upward) closure of a singleton is a residual.

1.3 . Measuring classical operations on wqos

Wqos are often obtained through operations on smaller wqos. In this thesis, we deal with classical binary and unary constructions, which we will call power operations. All these operations preserve well quasi-orderedness.

Binary operations.

For any wqos A, B, the disjoint sum, denoted with $A \sqcup B$, is defined as the order $\leq_\sqcup$ on $A \cup B$ (w.l.o.g. we suppose that this is a disjoint union), such that for all $x, y \in A \sqcup B$, $x \leq_\sqcup y$ iff $x \leq_A y$ or $x \leq_B y$.

The direct sum $\Sigma_{i<\alpha} A_i$ along an ordinal α is the order $\leq_+$ on the union of a family of wqos $(A_i)_{i<\alpha}$ (again we suppose that it is a disjoint union) such that for any $i, j < \alpha$ for any $x \in A_i, y \in A_j$, $x \leq_+ y$ iff $i < j$ or $i = j \wedge x \leq_{A_i} y$. If $\alpha = 2$ then $\Sigma_{i<\alpha} A_i$ can be written $A_0 + A_1$.

The Cartesian product $A \times B$ is ordered component-wise: for all $x_A, y_A \in A$ and $x_B, y_B \in B$, $(x_A, x_B) \leq_\times (y_A, y_B)$ iff $x_A \leq_A y_A$ and $x_B \leq_B y_B$.

The direct product, aka the lexicographic product, $A \cdot B$, is ordered by lexicographic order: $(x_A, x_B) \leq_{lex} (y_A, y_B)$ iff $x_B <_A y_B$, or $x_B \leq_B y_B$ and $x_A \leq_A y_A$. It is a direct sum of copies of A along B; it can be seen as B where each element has been replaced by a copy of A.

(NB: in a dictionary, the lexicographic order compare the letters left to right, in a wqo the priority is given to the right, to be consistent with the way we note ordinals. Indeed the ordinal product $\alpha \cdot \beta$ is a direct product for ordinals seen as wqos).

Power operations.

A^* denotes the star operation applied to a qo A, i.e., the set of finite words on the alphabet A ordered with the embedding order: For any finite words $u = u_1 \dots u_n$ and $v = v_1, \dots v_m$, $u \leq_{A^*} v$ iff there is an injective and increasing mapping $f : [1, n] \to [1, m]$ such that for all $i \in [1, n]$, $u_i \leq_A v_{f(i)}$.

We write $\mathcal{T}(A)$ for the collection of finite labelled ranked rooted trees on A. A tree is given as a tuple $x, t_1, \dots t_n$ with $x \in A$, $t_i \in \mathcal{T}(A)$ for all $1 \leq i \leq n$, $n \in \mathbb{N}$. If $n = 0$, then x is a leaf. $\mathcal{T}(A)$ is ordered by homeomorphic tree embedding $\leq_\mathcal{T}$:

$$(x, t_1, \dots t_n) \leq_\mathcal{T} (y, t'_1, \dots t'_m) \text{ iff } \begin{cases} (x, t_1, \dots t_n) \leq_\mathcal{T} t'_i \text{ for some } i \leq m, \text{ or} \\ x \leq_A y \text{ and } t_1 \dots t_n \leq_* t'_1 \dots t'_m \text{ in } (\mathcal{T}(A))^* . \end{cases}$$

We write $\mathsf{P_f}(A)$ for the collection of finite subsets of A endowed with the Hoare embedding (also known as the domination quasi-ordering) relation, stating that $S \sqsubseteq_{\mathcal{H}} S'$ whenever every element $x \in S$ is below some element $x' \in S'$ (see Example 1.1.1).

$M(A)$ denotes the set of finite multisets, or bags, of A, i.e., a generalisation of the notion of set where each element can appear finitely times. There are two main orderings classically defined on on the set of finite multisets. The most intuitive one is the multiset embedding on $M(A)$, defined by Weiermann (2009), also known as the term ordering, defined as:

$m \leq_\diamond m'$ iff there exists $f : m \to m'$ injective such that for any $x \in m$, $x \leq f(x)$.

Observe that $\mathsf{M}^\diamond(A)$ is isomorphic to A^* divided by the equivalence relation "the words u and v are equal modulo some permutation of letters". Therefore, according to Higman's lemma, $\mathsf{M}^\diamond(A)$ is a wqo when A is.

On the other hand, the multiset ordering $\mathsf{M}^\mathsf{r}(A)$, introduced by Dershowitz and Manna (1979) is less intuitive but better-known, especially in the rewriting community. It is usually defined on $M(A)$ where A is a well partial order as:

$$m \leq_r m' \iff m = m' \text{ or } \forall x \in m \backslash (m \cap m'), \exists y \in m' \backslash (m \cap m'), x <_A y \,.$$

Remark 1.3.1. I wrote earlier that one can easily turn a quasi-ordering into a partial ordering by quotienting it with the equivalence relation, and therefore it makes no difference if we deal with wqos instead of wpos. However, the wqos constructions that we study need to verify one desirable property: Let $C(A)$ be some wqo built from a wqo A, where C is some constructor. Then we want C to verify $C(A)_{/\equiv} \cong C(A_{/\equiv})_{/\equiv}$, i.e., C can be seen as applying to the related wpo.

All the constructions described earlier verify this property. However, the multiset ordering as defined on wpos does not, because the equality relation in a wqo, and the multiset subtraction $m \backslash m'$ are not defined modulo equivalence.

Example 1.3.2. *Let* $A = \{ a, b \}$ *with* $a \equiv b$. *Then* $\mathsf{M}^\mathsf{r}(A_{/\equiv}) = \mathsf{M}^\mathsf{r}(\{ a \})$ *which is a linear* wpo. *However the singleton multisets containing* a *and* b *are incomparable in* $\mathsf{M}^\mathsf{r}(A)$ *hence* $\mathsf{M}^\mathsf{r}(A)_{/\equiv}$ *is not linear.*

Therefore we need to tweak our definition of the multiset ordering to extend it to wqos: If A is a wqo, we define $\mathsf{M}^\mathsf{r}(A)$ as $\mathsf{M}^\mathsf{r}(A_{/\equiv_A})$. With this definition, M^r verifies our desirable property trivially, and $\mathsf{M}^\mathsf{r}(A)$ is always a wpo even when A is a wqo but not a wpo.

We can compute most of the ordinal invariants of these constructions compositionally. A survey on such invariants computation can be found in Džamonja et al. (2020): we recall in Table 1.1 the results that will be useful for us, using abbreviated notations defined in Figure 1.3.

Operation	M.O.T.	Height	Width
$A \sqcup B$	$\mathbf{o}(A) \oplus \mathbf{o}(B)$ [a]	$\max(\mathbf{h}(A), \mathbf{h}(B))$	$\mathbf{w}(A) \oplus \mathbf{w}(B)$ [c]
$A + B$	$\mathbf{o}(A) + \mathbf{o}(B)$	$\mathbf{h}(A) + \mathbf{h}(B)$	$\max(\mathbf{w}(A), \mathbf{w}(B))$
$A \times B$	$\mathbf{o}(A) \otimes \mathbf{o}(B)$ [a]	$\mathbf{h}(A) \hat{\oplus} \mathbf{h}(B)$ [c]	?
$A \cdot B$	?	$\mathbf{h}(A) \cdot \mathbf{h}(B)$	$\mathbf{w}(A) \odot \mathbf{w}(B)$ [c]
$\mathsf{M}^\diamond(A)$	$\omega^{\widehat{\mathbf{o}(A)}}$ [d]	$\mathbf{h}^*(A)$	$\mathbf{o}(\mathsf{M}^\diamond(A))$ if $\mathbf{o}(A)$ mult.indec.
$\mathsf{M}^\mathsf{r}(A)$	$\omega^{\mathbf{o}(A)}$ [e]	?	?
$A^*, A \neq \emptyset$	$\omega^{\omega^{(\mathbf{o}(A)^\pm)}}$ [b]	$\mathbf{h}^*(A)$	$\mathbf{o}(A^*)$ if $\mathbf{o}(A) \geq 2$
$\mathcal{T}(A)$	fct. of $\mathbf{o}(A)$ [b]	$\mathbf{h}^*(A)$	$\mathbf{o}(\mathcal{T}(A))$
$\mathsf{P_f}(A)$	?	?	?

Table 1.1: How to compute ordinal invariants compositionally.

Patterns in Table 1.1. Ignoring the unresolved cases, one can observe a pattern in Table 1.1: the ordinal invariants of a construction are expressed as a function of the ordinal invariants of the wqos taken in argument by the construction. We say that the ordinal invariants of a construction are functional (implicitly "in the ordinal invariants of A and B"). To be more thorough, we will distinguish the patterns of binary and power operations.

For binary constructions $C(A, B)$, each known ordinal invariant $\mathbf{f}(C(A, B))$ is a function of $\mathbf{f}(A)$ and $\mathbf{f}(B)$.

For power constructions $C(A)$, the situation is slightly more complicated, for instance the width of $\mathcal{T}(A)$ depends on $\mathbf{o}(A)$ and not $\mathbf{w}(A)$. Let us delve more into the patterns to understand why.

[a] de Jongh and Parikh (1977)
[b] Schmidt (1979)
[c] Abraham and Bonnet (1999)
[d] Weiermann (2009)
[e] Weiermann (1991); Van der Meeren et al. (2015)
All others: Džamonja et al. (2020)

Figure 1.3: Definition of the notations used in Table 1.1.

$$\alpha \mathbin{\hat{\oplus}} \beta \stackrel{\mathsf{def}}{=} \sup \{\, \alpha' \oplus \beta' \mid \alpha' < \alpha, \beta' < \beta \,\} \ .$$

$$\alpha^{\pm} \stackrel{\mathsf{def}}{=} \begin{cases} \alpha - 1 & \text{if } \alpha \text{ is finite,} \\ \alpha + 1 & \text{if } \alpha = \epsilon + n \text{ with } \omega^{\epsilon} = \epsilon \text{ and } n < \omega, \\ \alpha & \text{otherwise.} \end{cases}$$

$$\mathbf{h}^*(A) \stackrel{\mathsf{def}}{=} \begin{cases} \mathbf{h}(A) & \text{if } \mathbf{h}(A) \text{ is add. indec.,} \\ \mathbf{h}(A) \cdot \omega & \text{otherwise.} \end{cases}$$

$$\widehat{\alpha} \stackrel{\mathsf{def}}{=} \omega^{\alpha_1^\circ} + \cdots + \omega^{\alpha_n^\circ} \text{ when } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \ ,$$

$$\text{where } \alpha^{\circ} \stackrel{\mathsf{def}}{=} \begin{cases} \alpha + 1 & \text{if } \alpha = \epsilon + n \text{ with } \omega^{\epsilon} = \epsilon, \\ \alpha & \text{otherwise.} \end{cases}$$

For a power operation C, the mot $\mathbf{o}(C(A))$, when known, is a function of $\mathbf{o}(A)$, and multiplicatively indecomposable under some conditions: if $\mathbf{o}(A)$ is additively indecomposable for $\mathsf{M}^{\diamond}(A)$ and $\mathsf{M}^{\mathsf{r}}(A)$, if $\mathbf{o}(A) \geq 2$ for A^*. The mot of $\mathcal{T}(A)$, computed in Schmidt (1979) and expressed using Klammersymbols (a notation for large ordinals), is also multiplicatively indecomposable for any non-empty A (this is all we need to know for the purpose of this thesis).

Moreover, except for the multiset ordering, we always have $\mathbf{o}(A) < \mathbf{o}(C(A))$ (indeed, the definitions of $\alpha^{\pm}$ and α° deal separately with the case $\alpha = \epsilon + n$ to ensure that property). In fact you even have $\mathbf{o}(A)^{\omega} \leq \mathbf{o}(C(A))$.

However, if power constructions increase exponentially the mot, they leave height almost untouched: the height of $C(A)$, when known, is always equal to $\mathbf{h}^*(A)$, which can be alternatively defined as "the smallest additive indecomposable ordinal that is larger or equal than $\mathbf{h}(A)$", or you could say, $\mathbf{h}(A)$ rounded up. Which brings on some additional property: $\mathbf{h}(C(A)) < \mathbf{o}(C(A))$.

This is not innocuous, as we have the following relation between mot, width and height.

Theorem 1.3.3 (Heigh-Width Theorem Kříž and Thomas (1990b))**.** *For all wqo A, $\mathbf{o}(A) \leq \mathbf{w}(A) \otimes \mathbf{h}(A)$.*

Corollary 1.3.4 (Džamonja et al. (2020))**.** *For all wqo A, if $\mathbf{o}(A)$ is multiplicatively indecomposable, and $\mathbf{h}(A) < \mathbf{o}(A)$ then $\mathbf{w}(A) = \mathbf{o}(A)$.*

Proof. We know that $\mathbf{w}(A) \leq \mathbf{o}(A)$ (Lemma 1.2.5). If $\mathbf{w}(A) < \mathbf{o}(A)$, then $\mathbf{w}(A) \otimes \mathbf{h}(A) < \mathbf{o}(A)$ by definition of multiplicatively indecomposable, in contradiction with Theorem 1.3.3. Hence $\mathbf{w}(A) = \mathbf{o}(A)$. □

Therefore, when $\mathbf{o}(C(A))$ is multiplicatively indecomposable, we have $\mathbf{w}(C(A)) = \mathbf{o}(C(A))$.

Freund and Manca (2023) goes further in the analysis of the mot of power constructions, wondering why the mot of some power constructions ($\mathsf{M}^\diamond(A), A^*$) are not normal functions (strictly increasing and continuous at limits), whereas $\mathbf{o}(A) \mapsto \mathbf{o}(\mathsf{M}^\mathsf{r}(A))$ is.

Missing cases are missing for good reasons. All ordinal invariants in Table 1.1 are functional. As modularity is a good tool for tackling complexity, functionality is a desired feature: we want to be able to compute the ordinal invariants of a complex construction made of several nested operations only by looking at the ordinal invariants of base blocks, without having to "open the hood" and look at the wqo in more details. Therefore it would be bothersome if an invariant of a construction $C(A)$ was not functional, i.e. could not be expressed as a function of any ordinal invariants of A.

Now consider some of the unknown cases in Table 1.1. The mot of the direct product, the width of the cartesian product, the width of the multiset ordering, and all three ordinal invariants of the finite powerset have one thing in common: they are not functional.

To show that some invariant, for instance the width of the multiset ordering, is not functional, one only needs to exhibit two wqos X_1 and X_2 whose ordinal invariants are identical, and show that the width of $\mathsf{M}^\mathsf{r}(X_1)$ and $\mathsf{M}^\mathsf{r}(X_2)$ differ. As an apetizer, we will exhibit now such examples for the non-functional constructions cited above, and compute their ordinal invariants later.

The cornerstone of most of our non-functionality examples is $H \stackrel{\mathsf{def}}{=} \sum_{n<\omega} \Gamma_n$. Observe that for any $x \in H$, the residuals H_{*x} for $* \in \{ <, \perp, \not\geq \}$ are finite but can be arbitrarily large (see Figure 1.4). These residuals are represented on Figure 1.4 by the set of empty blue circles, the set of filled red circles, and the union of the blue and red sets, respectively. Therefore, by the method of residuals, $\mathbf{w}(H) = \mathbf{o}(H) = \mathbf{h}(H) = \omega$.

Example 1.3.5 (Non-functionality example: Cartesian product)**.** *Let* $X_1 \stackrel{\mathsf{def}}{=} H + \omega$ *and* $X_2 \stackrel{\mathsf{def}}{=} H + H$. *Then* X_1 *and* X_2 *have the same ordinal invariants. However* $\mathbf{w}(X_1 \times \omega) \neq \mathbf{w}(X_2 \times \omega)$.

Example 1.3.6 (Non-functionality example: Multiset ordering)**.** *As in Example 1.3.5, let* $X_1 \stackrel{\mathsf{def}}{=} H + \omega$ *and* $X_2 \stackrel{\mathsf{def}}{=} H + H$. *Then* $\mathbf{w}(\mathsf{M}^\mathsf{r}(X_1)) \neq \mathbf{w}(\mathsf{M}^\mathsf{r}(X_2))$.

Example 1.3.7 (Non-functionality example: Direct product)**.** $D_1 \stackrel{\mathsf{def}}{=} \Gamma_1 + \Gamma_2$ *and* $D_2 \stackrel{\mathsf{def}}{=} \Gamma_2 + \Gamma_1$ *have the same ordinal invariants, but* $\mathbf{o}((\omega + 1) \cdot D_1) \neq \mathbf{o}((\omega + 1) \cdot D_2)$.

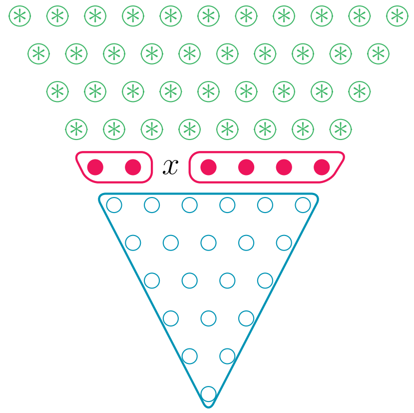


Figure 1.4: Residuals of H.

Example 1.3.8 (Non-functionality example: Finite powerset)**.** *Consider* $Y_1 = (\omega+\omega) \sqcup (\omega+\omega)$ *and* $Y_2 = (\omega \sqcup \omega) + (\omega \sqcup \omega)$*. These two* wqos *have the same ordinal invariants, but* $\mathsf{P_f}(Y_1)$ *and* $\mathsf{P_f}(Y_2)$ *differ on all three ordinal invariants.*

We do not yet have all the tools to compute the ordinal invariants in these examples. We will remedy this in the next chapter.

2 - Toolbox

This chapter presents several useful tools and lemmas to compute ordinal invariants.

Section 2.1 introduces well-known notions of relations to compare wqos. Section 2.2 presents a game-and-strategies approach to computing ordinal invariants, which leads to the development of quasi-incomparable subsets in Section 2.3. Section 2.4 is a direct application of the result Section 2.3. Finally, Section 2.5 investigates what we can deduce on the structure of a wqo knowing its ordinal invariants.

2.1 . Wqos relations

Ordinal invariants are monotonic with respect to some relations between wqos.

Definition 2.1.1 (Reflection)**.** A mapping $f\colon A \to B$ is a *reflection* if $f(x) \leq_B f(y)$ implies $x \leq_A y$. When there is a reflection from A to B, we note $A \hookrightarrow B$.

Remark 2.1.2. The term reflection was coined by Bonnet et al. (2013), but is actually a rediscovery of the notion of *quasi-embeddings*(Weiermann, 2009). They proved a relation between reflection and the simulation with respect to coverability languages for wsts.

A reflection with monotonicity is called an embedding. We prefer to use the related notion of substructure (Džamonja et al., 2020).

Definition 2.1.3 (Substructure)**.** A quasi-order $(A, \leq_A)$ is a *substructure* of a quasi-order $(B, \leq_B)$ whenever $A \subseteq B$ and $\leq_A$ is the restriction of $\leq_B$ to A. This relation is written $A \leq_{\mathsf{st}} B$.

Fraïssé (1986) introduces the notion of reinforcements, sometimes called extensions, which corresponds to the notion of augmentations in Džamonja et al. (2020).

Definition 2.1.4 (Augmentation)**.** A quasi-order $(A, \leq_A)$ is an *augmentation* of a quasi-order $(B, \leq_B)$ whenever $A = B$ and $\leq_B \subseteq \leq_A$. We write this relation $B \leq_{\mathsf{aug}} A$.

Note that if $B \leq_{\mathsf{aug}} A$ or $A \leq_{\mathsf{st}} B$, then $A \hookrightarrow B$. Moreover, if there is a reflection $f\colon A \to B$, then $A \geq_{\mathsf{aug}} f(A) \leq_{\mathsf{st}} B$.

We use implicitly these notions modulo quasi-isomorphism. In particular, for the augmentation, we write $B \leq_{\mathsf{aug}} A$ to mean that $B_{/\equiv_B} \leq_{\mathsf{aug}} A$. Why not $A_{/\equiv_A} \geq_{\mathsf{aug}} B_{/\equiv_B}$? Take for instance $A, B = \{ a, b \}$ ordered with $a \equiv_A b$ and $a \perp_B b$. Then $A \geq_{\mathsf{aug}} B$ but $A_{/\equiv_A} \leq_{\mathsf{st}} B_{/\equiv_B}$.

These relations allow us to compare the ordinal invariants of different wqos.

Lemma 2.1.5. *For any* wqos A, B*:*

- *If* $A \leq_{\mathsf{st}} B$ *then* $\mathbf{f}(A) \leq \mathbf{f}(B)$ *for* $\mathbf{f} \in \{ \mathbf{o}, \mathbf{w}, \mathbf{h} \}$.
- *If* $A \geq_{\mathsf{aug}} B$ *then* $\mathbf{f}(A) \leq \mathbf{f}(B)$ *for* $\mathbf{f} \in \{ \mathbf{o}, \mathbf{w} \}$*. Moreover, if* A *is a* wpo *then* $\mathbf{h}(A) \geq \mathbf{h}(B)$.
- *If* $A \hookrightarrow B$ *then* $\mathbf{f}(A) \leq \mathbf{f}(B)$ *for* $\mathbf{f} \in \{ \mathbf{o}, \mathbf{w} \}$.

We can now translate Lemma 1.2.10 and Theorems 1.2.6 and 1.2.7 in terms of augmentations and substructures :

Lemma 2.1.6. *For any ordinals* $\beta \leq \alpha$*, any* $\mathbf{f} \in \{ \mathbf{o}, \mathbf{w}, \mathbf{h} \}$*, and any wqo* A *such that* $\mathbf{f}(A) = \alpha$*, there exists* $B \leq_{\mathsf{st}} A$ *such that* $\mathbf{f}(B) = \beta$.

Proof. If $\beta = \alpha$ then let $B = A$. Otherwise, according to Lemma 1.2.10, for any $\beta < \mathbf{o}(A)$ (resp. $\mathbf{w}(A)$, $\mathbf{h}(A)$), there exists a bad sequence (resp. an antichain, a decreasing sequence) s of A such that $\mathbf{o}(A_{\not\geq s}) = \beta$ (resp. $\mathbf{w}(A_{\perp s}) = \beta$, $\mathbf{h}(A_{<s}) = \beta$). We can take B such a residual, since a residual of A is a substructure of A. □

Definition 2.1.7. A wpo A is a *linearisation* of a wqo B if A is linear and $B \leq_{\mathsf{aug}} A$. Since an augmentation is defined modulo quasi-isomorphism, there is a function $f : B \to A$ such that $x \leq_A y$ implies $f(x) \leq_B f(y)$. We call f a *linearisation function* of A.

Lemma 2.1.8 (Minimax theorems reformulated)**.** *For any* wqo A*,* $\mathbf{o}(A)$ *is the largest ordinal* α *such that* $\alpha \geq_{\mathsf{aug}} A$*. Similarly,* $\mathbf{h}(A)$ *is the largest ordinal* β *such that* $\beta \leq_{\mathsf{st}} A$.

Let us now see how classical constructions relate to each other with respect to these relations.

We say that a wqo construction $C(A_1, \dots A_n)$ is monotonic with respect to some relation $\leq$ when $A_i \leq B_i$ for all $i \in [1, n]$ implies that $C(A_1, \dots A_n) \leq C(B_1, \dots B_n)$.

Proposition 2.1.9 (Monotonicity of classical constructions)**.** *All the* wqo *constructions defined in Section 1.3 are monotonic with respect to* $\hookrightarrow$*,* $\leq_{\mathsf{aug}}$ *and* $\leq_{\mathsf{st}}$*, with the exception of* $\mathsf{M}^{\mathsf{r}}(A)$ *that is always monotonic with respect to* $\hookrightarrow$ *and* $\leq_{\mathsf{st}}$*, but only monotonic with respect to* $\leq_{\mathsf{aug}}$ *if we restrict ourselves to* wpos.

Lemma 2.1.10 (Wqo alchemy)**.** *Let A, B, C be* wqos*.*

$$A \times (B \sqcup C) = A \times B \sqcup A \times C \;, \tag{Alc-X-1}$$
$$A \times (B + C) \leq_{\mathsf{aug}} A \times B + A \times C \;, \tag{Alc-X-2}$$
$$A \cdot (B \sqcup C) = A \cdot B \sqcup A \cdot C \;, \tag{Alc-D-1}$$
$$A \cdot (B + C) = A \cdot B + A \cdot C \;, \tag{Alc-D-2}$$
$$M^*(A \sqcup B) = M^*(A) \times M^*(B) \text{ for } * \in \{\diamond, r\} \;, \tag{Alc-M-1}$$
$$\mathsf{M}^{\mathsf{r}}(A + B) = \mathsf{M}^{\mathsf{r}}(A) \cdot \mathsf{M}^{\mathsf{r}}(B) \;, \tag{Alc-M-2}$$
$$\mathsf{M}^{\diamond}(A + B) \leq_{\mathsf{aug}} \mathsf{M}^{\diamond}(A) \cdot \mathsf{M}^{\diamond}(B) \;, \tag{Alc-M-3}$$
$$\mathsf{P_f}(A \sqcup B) \cong \mathsf{P_f}(A) \times \mathsf{P_f}(B) \;, \tag{Alc-Pf-1}$$
$$\mathsf{P_f}(A + B) \cong \mathsf{P_f}(A) + (\mathsf{P_f}(B) \setminus \{\, \emptyset \,\}) \;. \tag{Alc-Pf-2}$$

Proof. Equation (Alc-X-1) For all $a, a' \in A, p, p' \in (B \sqcup C)$, $(a, p) \leq (a', p')$ in $A \times (B \sqcup C)$ iff $a \leq_A a'$ and $p \leq_{B \sqcup C} p'$. And $p \leq_{B \sqcup C} p'$ iff p, p' belong both to B and $p \leq_B p'$, or they belong both to C and $p \leq_C p'$. Hence $(a, p) \leq (a', p')$ in $A \times (B \sqcup C)$ iff $(a, p) \leq (a', p')$ in $A \times B \sqcup A \times C$.

Equation (Alc-X-2) For all $a, a' \in A, p, p' \in (B \cup C)$, $(a, p) \leq (a', p')$ in $A \times (B + C)$ iff $a \leq_A a'$ and $p \leq_{B+C} p'$. And $p \leq_{B+C} p'$ iff either p and p' belong both to B and $p \leq_B p'$, or they belong both to C and $p \leq_C p'$, or $p \in B$ and $p' \in C$. Hence $(a, p) \leq (a', p')$ in $A \times (B + C)$ iff $(a, p) \leq (a', p')$ in $A \times B + A \times C$.

Equation (Alc-M-1) For $* \in \{\diamond, r\}$, for any $m, m' \in M(A+B)$, $m = m_A \cup m_B$ and $m' = m'_A \cup m'_B$ for some $m_A, m'_A \in M(A)$, $m_B, m'_B \in M(B)$. Then $m \leq_{M^*(A \sqcup B)} m'$ iff $m_A \leq_{M^*(A)} m'_A$ and $m_B \leq_{M^*(B)} m'_B$.

Equation (Alc-M-2) Assume A and B are wpos. Now $(m_A, m_B) \leq_{\mathsf{M}^{\mathsf{r}}(A) \cdot \mathsf{M}^{\mathsf{r}}(B)} (m'_A, m'_B)$ iff $m_B <_{\mathsf{M}^{\mathsf{r}}(B)} m'_B$ or $m_B = m'_B$ and $m_A \leq_{\mathsf{M}^{\mathsf{r}}(A)} m'_A$. If $m_B <_{\mathsf{M}^{\mathsf{r}}(B)} m'_B$ then $m <_{\mathsf{M}^{\mathsf{r}}(A+B)} m'_B \leq_{\mathsf{M}^{\mathsf{r}}(A+B)} m'$. If $m_B = m'_B$ then $m \setminus (m \cap m') = m_A \setminus (m_A \cap m'_A)$ and $m' \setminus (m \cap m') = m'_A \setminus (m_A \cap m'_A)$, hence $m \leq_{\mathsf{M}^{\mathsf{r}}(A+B)} m' \iff m_A \leq_{\mathsf{M}^{\mathsf{r}}(A)} m'_A$.

Equation (Alc-M-3) Suppose that $m \leq_{\mathsf{M}^{\diamond}(A+B)} m'$. By definition there exists an injective function $f : m \to m'$ such that $x \leq_{A+B} f(x)$ for any $x \in m$. Then $f(m_B) \subseteq m'_B$ hence $m_B \leq_{\mathsf{M}^{\diamond}(B)} m'_B$. If $m_B <_{\mathsf{M}^{\diamond}(B)} m'_B$ then $(m_A, m_B) <_{\mathsf{M}^{\diamond}(A) \cdot \mathsf{M}^{\diamond}(B)} (m'_A, m'_B)$. Otherwise $m_B = m'_B$. Then $f(m_A) \subseteq m'_A$, hence $m_A \leq_{\mathsf{M}^{\diamond}(A)} m'(A)$. Thus $(m_A, m_B) \leq_{\mathsf{M}^{\diamond}(A) \cdot \mathsf{M}^{\diamond}(B)} (m'_A, m'_B)$.

Equation (Alc-Pf-1) For all $S, S' \in \mathsf{P_f}(A \sqcup B)$, let $S_A = S \cup A$, $S_B = S \cup B$, $S'_A = S' \cup A$ and $S'_B = S' \cup B$. Then $S \sqsubseteq_{\mathcal{H}} S'$ iff $S_A \sqsubseteq_{\mathcal{H}} S'_A$ and $S_B \sqsubseteq_{\mathcal{H}} S'_B$.

Equation (Alc-Pf-2) We consider $\iota : S = S_A \cup S_B \mapsto \begin{cases} S_B \text{ if } S_B \neq \emptyset \\ S_A \text{ otherwise.} \end{cases}$
Observe that ι is a bijection from $\mathsf{Ant}(A+B)$ into $\mathsf{Ant}(A) + (\mathsf{Ant}(B) \setminus \{ \emptyset \})$. Moreover, for all $S, S' \in \mathsf{P_f}(A+B)$, $S \sqsubseteq_{\mathcal{H}} S'$ iff $\iota(S) \leq \iota(S')$ in $\mathsf{P_f}(A) + (\mathsf{P_f}(B) \setminus \{ \emptyset \})$.

□

2.2 . Games

We present here an alternative definition of ordinal invariants, that will be more intuitive to game theory-minded readers. The game we present here deals with width and antichains, but it can be easily transposed to mot and bad sequences, or height and decreasing sequences.

Definition 2.2.1. For all wqo A and ordinal α, $G_{A,\alpha}$ is a game for two players, let's call them Antoine (for antichain) and Odile (for ordinal), with the following rules:

- Each configuration of the game is a pair (Y, γ) with Y an antichain of A, and $\gamma \leq \alpha$.
- The game begins in $(\emptyset, \alpha)$. Either Odile or Antoine begins.
- At Odile's turn, she picks some $\gamma' < \gamma$ and moves from configuration (Y, γ) to (Y, γ').
- At Antoine's turn, he picks some $x \in A_{\perp Y}$ and moves from configuration (Y, γ) to $(Y \cup \{x\}, \gamma)$.
- The first player who cannot play loses.

This game is a specific case of the games defined in Blass and Gurevich (2008) and Džamonja et al. (2020). Since A is FAC and α is WF, the players cannot play forever, so the game terminates.

Lemma 2.2.2 (Blass and Gurevich (2008),Džamonja et al. (2020))**.** $\mathbf{w}(A) \leq \alpha$ *iff Odile has a winning strategy when Antoine starts the game.* $\mathbf{w}(A) \geq \alpha$ *iff Antoine has a winning strategy when Odile starts the game.*

Intuitively, one can see this game as playing along a branch of $\mathsf{Inco}(A)$: each time Antoine plays $Y \leftarrow Y \cup \{x\}$, he moves from node Y to its child $Y \cup \{x\}$. Odile has a winning strategy when she can play $\gamma \geq r(Y)$. Antoine has a winning strategy when he can play Y such that $\gamma \leq r(Y)$.

Therefore, to prove $\mathbf{w}(A) = \alpha$, we only need to exhibit two winning strategies, one for each player depending on who starts the game.

Example 2.2.3. *Let us play on $\mathbb{N} \times \mathbb{N}$. Odile starts with ω. She picks some $n < \omega$. Antoine starts his antichain with (n, n). Then Odile will play $n - 1, n - 2, \ldots$ until 0, hence she only have n turns left to play, whereas Antoine can build the antichain $(n, n), (n-1, n+1), \ldots (0, 2n)$ in n turns. Thus he can outlast her. Antoine wins, proving that $\mathbf{w}(\mathbb{N} \times Nat) \geq \omega$.*

Now imagine that Antoine starts, with some element (n, m). Odile can just look up the rank $r((n, m)) = n + m$, and she picks the ordinal $n + m$. One maximal antichain that Antoine can play is the anti-diagonal $(0, n+m, \ldots (n-1, m+1), (n, m), (n+1, m-1), \ldots, (n+m, 0)$, and thus he last $n + m$ more turns. Odile can outlast him by decreasing by 1 at each turn. Odile wins, proving that $\mathbf{w}(\mathbb{N} \times \mathbb{N}) \leq \omega$.

Games allow us to give more intuitive proofs based on reasoning about strategies, which would become tedious and inscrutable when translated in terms of trees and rank. For the next lemma, we give you two proofs of the same lemma so you can compare the two techniques.

Remark 2.2.4. For any quasi-order A, when we take a subset B of A, we always imply that B is ordered with $\leq_A$ restricted to B. Thus a subset of a wqo is a wqo.

Lemma 2.2.5 (How to combine winning strategies for Odile)**.** *Let $A = \cup_{i \leq n} A_i$ a set partitioned into n subsets, for some $n \in \mathbb{N}$. Let $\leq_A$ a well-partial ordering on A, and $\leq_{A_i}$ the same ordering restricted to the subset A_i for $i \leq n$. Then for any ordinal invariant $\mathbf{f} \in \{ \mathbf{o}, \mathbf{w}, \mathbf{h} \}$,*

$$\mathbf{f}(A, \leq_A) \leq \bigoplus_{i \leq n} \mathbf{f}(A_i, \leq_{A_i}) \ .$$

This lemma can be found in Delhommé (2006) for $\mathbf{f} = \mathbf{h}$, with yet another proof.

Proof by induction on rank in trees. Let us do the proof for $\mathbf{f} = \mathbf{o}$. From any bad sequence s on A, one can extract a bad sequence s_i by restricting s to A_i for any $i \leq n$. Let rk be the rank function on $\mathsf{Bad}(A)$, and rk_i the rank function on $\mathsf{Bad}(A_i)$ for any $i \leq n$.

By induction on the rank of s in $\mathsf{Bad}(A)$, one shows that $rk(s) \leq \bigoplus_{i \leq n} rk_i(s_i)$. This is trivially true for $rk(s) = 0$. Now assume that this property is true for any bad sequence of rank inferior to $rk(s)$.

$$\begin{aligned}
rk(s) &= \sup_{x \in A_{\not\geq s}} \{rk(sx) + 1\} \\
&\leq \sup_{1 \leq i \leq n} \left(\sup_{x \in (A_i)_{\not\geq s_i}} \{rk(sx) + 1\} \right) \\
&\overset{\text{(IH)}}{=} \sup_{1 \leq i \leq n} \left(\sup_{x \in (A_i)_{\not\geq s_i}} \left\{ \mathrm{rk}(s_i x) \oplus \bigoplus_{j \leq n, j \neq i} \mathrm{rk}(s_j) + 1 \right\} \right) \\
&\leq \sup_{1 \leq i \leq n} \left(\sup_{x \in (A_i)_{\not\geq s_i}} \{\mathrm{rk}(s_i x)\} \oplus \bigoplus_{j \leq n, j \neq i} \mathrm{rk}(s_j) + 1 \right) \\
&= \sup_{1 \leq i \leq n} \bigoplus_{j \leq n} rk(s_j) \\
&= \bigoplus_{i \leq n} rk(s_i) \ .
\end{aligned}$$

□

Proof combining strategies. Let us do the proof for $\mathbf{f} = \mathbf{w}$ (this kind of reasoning can easily be translated to the other ordinal invariants).

Let $\alpha = \bigoplus_{i \leq n} \mathbf{w}(A_i, \leq_{A_i})$ and for each $1 \leq i \leq n$, let $\alpha_i = \mathbf{w}(A_i, \leq_{A_i})$, and let S_i be the winning strategy for Odile for G_{A_i,α_i} when Antoine begins. To prove that $\mathbf{w}(A, \leq_A) \leq \alpha$, we will exhibit a winning strategy for Odile for $G_{A,\alpha}$ when Antoine begins by combining the strategies S_i.

From any antichain s on A, one can extract n antichains s_i by restricting s to A_i for any $1 \leq i \leq n$.

Our invariant: Antoine has selected an antichain s, Odile has chosen in response the ordinal $\gamma \overset{\text{def}}{=} \bigoplus_{i \leq n} rk(s_i)$. It is Antoine's turn to play.

This invariant is trivially true at the beginning of the game.

At Antoine's turn, he selects $x \in A$ such that sx is an antichain. W.l.o.g assume that $x \in A_1$. Then $rk(s_1, x) < rk(s_1)$ hence

$$\gamma' \overset{\text{def}}{=} rk(s_1, x) \ \oplus \bigoplus_{2 \leq i \leq n} rk(s_i) < \gamma \ ,$$

thus Odile can move to configuration (sx, γ'), preserving the invariant.

Under this invariant, Antoine stops being able to play just after Odile reaches 0, thus Odile wins. □

2.3 . Quasi-incomparable subsets

Here we introduce a method to combine several winning strategies for Antoine in order to prove lower bounds on the width of complex wqos.

We write $A_1 \perp A_2$ with A_1 and A_2 two subsets of a wqo A when for any $a \in A_1, b \in A_2$, $a \perp_A b$. We say $A_1, \ldots, A_m$ is a family of incomparable subsets of A when $A_i \perp A_j$ for any $i \neq j$. Observe that, for such a family, we have $A \geq_{\mathsf{st}} \bigsqcup_i A_i$ thus $\mathbf{w}(A) \geq \bigoplus_i \mathbf{w}(A_i)$. However, we can do almost as well with a weaker condition on the family (A_i).

Definition 2.3.1 (Quasi-incomparable family of subsets.)**.** $A_1, \ldots, A_m$ is a family of *quasi-incomparable* subsets of a wqo A if for any $i \in [1, m]$, for every finite set $Y \subseteq A_1 \cup \cdots \cup A_{i-1}$, there exists $A'_i \subseteq A_i$ such that $\mathbf{w}(A'_i) = \mathbf{w}(A_i)$ and $A'_i \perp Y$.

Note that this notion is sensitive to the way we number the family (A_i).

Lemma 2.3.2 (How to combine winning strategies for Antoine)**.** *Let $A_1, \ldots, A_m$ be a family of quasi-incomparable subsets of A. Then* $\mathbf{w}(A) \geq \mathbf{w}(A_m) + \cdots + \mathbf{w}(A_1)$.

Proof. Let us note α_i for $\mathbf{w}(A_i)$

For any $i \in [1, m]$, for any subset A'_i of A_i such that $\mathbf{w}(A'_i) = \alpha_i$, Antoine has a winning strategy on $G_{A'_i, \alpha_i}$ when Odile begins. We want to combine those strategies into a winning strategy for Antoine on $G_{A, \alpha_m + \cdots + \alpha_1}$ when Odile begins.

Intuitively, the game is played in m phases. Odile goes through the sum $\alpha_m + \cdots + \alpha_1$ from right to left, which means that at the j-th phase she is decreasing the term α_j, while Antoine plays his strategie on A'_j the subset of A_j given by the quasi-incomparability definition, with Y the antichain built by Antoine during phases 1 to $j - 1$.

More formally, assume that Odile has selected some ordinal γ, and Antoine has selected an antichain $Y \subseteq A$. Now γ can be written in a unique way as $\gamma = \alpha_m + \cdots + \alpha_{k+1} + \sigma$ with $\sigma < \alpha_k$ for some $k \in [1, m]$: we say the game is in phase k.

Our invariant: During phase k, $Y \in A_1 \cup \cdots \cup A_k$. By definition of quasi-incomparable subsets, there exists $A'_k \subseteq A_k$ such that $\mathbf{w}(A'_k) = \alpha_k$ and $A'_k \perp (Y \cap (A_1 \cup \cdots \cup A_{k-1})$. The antichain $Y \cap A_k$ comes from the winning strategy S_k for Antoine for $G_{A'_k, \alpha_k}$.

Now it is Odile's turn, and she selects some $\gamma' < \gamma$. We know γ' is either above or strictly below $\alpha_m + \cdots + \alpha_{k+1}$:

- If $\gamma = \alpha_m + \cdots + \alpha_{j+1} + \sigma'$ with $j < k$ and $\sigma' < \alpha_j$ then we move to the jth phase of the game: By definition of quasi-incomparable subsets, there exists $A'_j \subseteq A_j$ isomorphic to A_j such that $A'_j \perp Y$. We follow the winning strategy S_j for Antoine for $G_{A'_j, \alpha_j}$ which selects some $x \in A'_j$. Since $A'_j \perp Y$, $Y \leftarrow Y \cup \{x\}$ is still an antichain.

- If $\gamma = \alpha_m + \cdots + \alpha_{k+1} + \sigma'$ for some $\sigma' < \sigma$, then we can keep applying the strategy S_k on A'_k, which selects some $x \perp Y \cap A'_k$. Since $A'_k \perp (Y \setminus A_k)$, $Y \leftarrow Y \cup \{x\}$ is still an antichain.

□

We often use a more restrictive definition of quasi-incomparable subsetss:

Definition 2.3.3 (Quasi-incomparable* subsets.)**.** $A_1, \ldots, A_m$ is a family of *quasi-incomparable** subsets of A if for any $i \in [1, m]$, for every finite set $Y \subseteq A_1 \cup \cdots \cup A_{i-1}$, there exists $A'_i \subseteq A_i$ isomorphic to A_i such that $A'_i \perp Y$.

Obviously, quasi-incomparable* subsets are quasi-incomparable, thus Lemma 2.3.2 stands.

Lemma 2.3.2 will be a cornerstone for measuring the Cartesian product (Chapter 4) and finite multiset (Chapter 5), but we can already use it to prove a nifty lower bound for the width of the Cartesian product applied to self-residual wqos.

2.4 . Lower bound for self-residual wqos

Definition 2.4.1 (Self-residual)**.** Let A be a quasi-order. Then A is *self-residual* if for any $x \in A$, $A_{\not\leq x}$ contains an isomorphic copy of A as a substructure.

A more precise name would be "self-residual for $\not\leq$", but we opt for shortness.

Remark 2.4.2. If A is self-residual, then for all finite $Y \subseteq A$, $A_{\not\leq Y}$ contains an isomorphic copy of A (by induction on the size of Y).

The idea for self-residuality is derived from the notion of transferability found in Džamonja et al. (2020):

Definition 2.4.3. A wqo A is *transferable* if $\mathbf{w}(A_{\not\leq Y}) = \mathbf{w}(A)$ for any finite $Y \subseteq A$.

As you can see, transferability is weaker than self-residuality. In practice, we observed that we often prove transferability by proving self-residuality, which led us to introduce this notion. These two definitions complement each other in the same way as the two definitions of quasi-incomparable subsets do.

Like transferability, the notion of self-residual is compatible with th Cartesian product. If A and B are self-residual qos, then $A \times B$ is self-residual.

Example 2.4.4. *Any additively indecomposable ordinal* α *is self-residual: for any* $x < \alpha$, $\alpha_{\not\leq x} \cong \alpha - (x+1) = \alpha$, *because* $x+1 < \alpha$, *through Equation* (A.7). *Furthermore, the Cartesian product of additively indecomposable ordinals is self-residual.*

Here is an application of Lemma 2.3.2 that will be useful in Section 4.2. Let us write $B \cdot k$ for the direct sum of k copies of B.

Lemma 2.4.5. *Let* A, B *be two wqos such that* A *is self-residual. Then* $\mathbf{w}(A \times (B \cdot k)) \geq \mathbf{w}(A \times B) \cdot k$.

Proof. Let $B_1, \ldots, B_k$ be disjoint copies of B, so that $B \cdot k = B_k + \cdots + B_1$. We claim that $(A \times B_i)_{i \in [1,k]}$ is a family of quasi-incomparable* subsets of $A \times (B \cdot k)$:

Fix $j \in [1, k-1]$ and $Y \subset (A \times B_1) \cup \cdots \cup (A \times B_j)$ finite (Figure 2.1 illustrates the case $j = 2$). We want to find a subset C of $A \times B_{j+1}$ isomorphic to $A \times B_{j+1}$ such that $C \perp Y$. Let $proj_A(Y) \stackrel{\mathsf{def}}{=} \{a \in A \mid (a, b) \in Y \text{ for some } b \in B_1 \cup \cdots \cup B_j\}$. Since A is self-residual, $A_{\not\leq proj_A(Y)}$ contains an isomorphic copy A' of A, hence $A' \times B_{j+1}$ is isomorphic to $A \times B_{j+1}$. For any $(a, b) \in A_{\not\leq proj_A(Y)} \times B_{j+1}$ and $(a', b') \in Y$, we know that $a \not\leq_A a'$ and $b <_{B \cdot k} b'$, thus $(a, b) \perp (a', b')$. Hence $A' \times B_{j+1} \perp Y$.

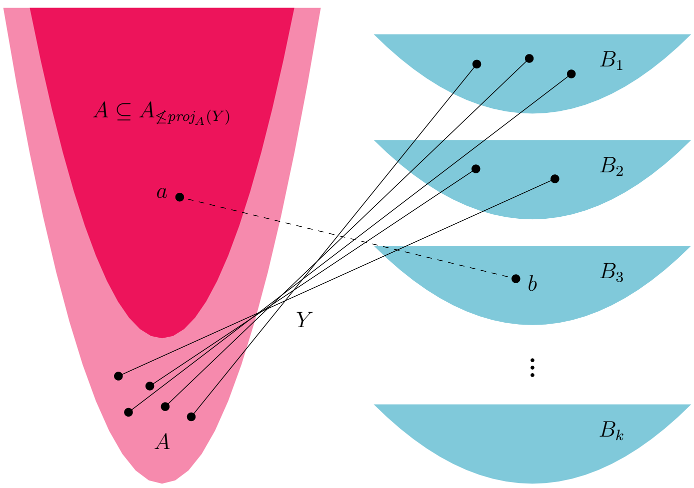


Figure 2.1: All elements (a, b) of $A_{\not\leq proj_A(Y)} \times (B_k + \cdots + B_3)$ are incomparable with $Y \subseteq A \times (B_2 + B_1)$.

Therefore $(A \times B_i)_{i \in [1,n]}$ is a family of quasi-incomparable* subsets of $A \times (B \cdot k)$, so according to Lemma 2.3.2,

$$\mathbf{w}\left(A \times \left(\sum_{i=k}^{1} B_i\right)\right) \geq \sum_{i=k}^{1} \mathbf{w}(A \times B_i) = \mathbf{w}(A \times B) \cdot k\ .$$

□

Remark 2.4.6. When $\mathbf{w}(A \times B)$ is additively indecomposable, this lower bound is tight: $A \times (B \cdot k) \geq_{\mathsf{aug}} A \times B \times \Gamma_k$, so $\mathbf{w}(A \times (B \cdot k)) \leq \mathbf{w}(A \times B) \otimes k = \mathbf{w}(A \times B) \cdot k$ through Lemma A.0.11.

2.5 . Structural lemmas

One can deduce some informations on the structure of a wqo A from the form of its ordinal invariants (e.g., are they successor or limit ordinals? finite or infinite? indecomposable?). This will prove useful in Chapter 6 to find bounds on the ordinal invariants of the finite powerset.

Lemma 2.5.1 (Mot: structural lemma)**.** *Let $(A, \leq)$ be a* wqo *such that* $\mathbf{o}(A) = \alpha + \beta$ *for some ordinals* α, β*. Then there exists a partition* [a] $A = A_\alpha \uplus A_\beta$ *such that:* $\mathbf{o}(A_\alpha) = \alpha$*,* $\mathbf{o}(A_\beta) = \beta$*, and* $A_\alpha + A_\beta \geq_{\mathsf{aug}} A \geq_{\mathsf{aug}} A_\alpha \sqcup A_\beta$*.*

This is a generalisation of Theorem 3.2 of de Jongh and Parikh (1977), which treats the case $\beta = 1$.

Proof. There exists a reflection $f : \alpha + \beta \to A$. We let $A_\alpha = f([0, \alpha[)$ and $A_\beta = f([\alpha, \alpha+\beta[)$. We further assume that f makes $\mathbf{o}(A_\alpha)$ minimal; since the collection of all reflections from $\alpha+\beta$ to A is a set, the collection of all possible $\mathbf{o}(A_\alpha)$ is a set of ordinals, which admits a minimal element. f gives a reflection from α to A_α, and one from β to A_β, hence $\alpha \leq \mathbf{o}(A_\alpha)$ and $\beta \leq \mathbf{o}(A_\beta)$. On the other hand, since f is a reflection, for all $a \in A_\alpha, b \in A_\beta$, $a \not\geq b$. It follows that

$$A_\alpha + A_\beta \geq_{\mathsf{aug}} A \geq_{\mathsf{aug}} A_\alpha \sqcup A_\beta \ ,$$

which implies that $\mathbf{o}(A_\alpha)+\mathbf{o}(A_\beta) \leq \alpha+\beta$ according to Lemma 2.1.5 and Table 1.1. Therefore $\mathbf{o}(A_\alpha) + \mathbf{o}(A_\beta) = \alpha + \mathbf{o}(A_\beta) = \alpha + \beta$. Therefore $\mathbf{o}(A_\beta) = \beta$ because the ordinal sum is left-cancellative.

Besides, since $\alpha \leq \mathbf{o}(A_\alpha)$, let $\gamma = \mathbf{o}(A_\alpha) - \alpha$. Since $\mathbf{o}(A_\alpha) + \beta = \alpha + \beta$, we must have $\gamma + \beta = \beta$.

There exists a reflection $g : \mathbf{o}(A_\alpha) = \alpha + \gamma \to A_\alpha$. We consider the reflection $f' : \alpha + \gamma + \beta \to A$ obtained by concatenating g and f restricted to A_β. If $\gamma \neq 0$, $A'_\alpha \stackrel{\mathsf{def}}{=} f'([0, \alpha[)$ is a strict subset of A_α, such that $\mathbf{o}(A'_\alpha) \geq \alpha$. Since $\mathbf{o}(A'_\alpha) < \mathbf{o}(A_\alpha)$, it follows that $\mathbf{o}(A_\alpha)$ is not minimal, which is absurd. Therefore, $\mathbf{o}(A_\alpha) = \alpha$. □

Lemma 2.5.2 (Height: structural lemma)**.** *Let A be a* wqo *such that* $\mathbf{h}(A) = \alpha + \beta$*. There exists a partition* $A = A_\bot \uplus A_\top$ *such that* $\mathbf{h}(A_\bot) = \alpha$ *and* $\mathbf{h}(A_\top) = \beta$*.*

[a] We consider a generalized definition of partition where subsets can be empty.

The following proof uses a technique called M-decomposition in Wolk (1967), a decomposition of A where elements are sorted according to their rank in $\mathsf{Dec}(A)$.

Proof. Without loss of generality, let us assume that A is a partial order. Let $r : \mathsf{Dec}(A) \to \mathbf{h}(A)$ be the rank function defined on $Dec(A)$. There is an antichain $s = \{ x \in A \mid r(x) = \alpha \}$. Let $A_\perp = \bigcup_{x \in s} A_{<x}$ and $A_\top = A \setminus A_\perp$. For all $x \in s$, $\mathbf{h}(A_{<x}) = \alpha$ (by definition of rank), hence $\mathbf{h}(A_\perp) \geq \alpha$. If $\mathbf{h}(A_\perp) > \alpha$ then there exists $x \in A_\perp$ of rank α according to Lemma 1.2.10, which is absurd. Similarly, $\mathbf{h}(A_\top) \geq \beta$, but if $\mathbf{h}(A_\top) = \gamma > \beta$, then there exists a chain $\gamma \leq_{\mathsf{st}} A_\top$ according to Lemma 2.1.8. W.l.o.g. the minimal element of this chain is some $x \in s$ (because s is the set of the minimal elements of $A_\top$ by construction). Since $\mathbf{h}(A_{<x}) = \alpha$ there exists a chain $\alpha \leq_{\mathsf{st}} A_{<x}$. Therefore $\alpha + \gamma \leq_{\mathsf{st}} A$ which is absurd since $\mathbf{h}(A) = \alpha + \beta < \alpha + \gamma$ (Equation (A.1)) and $\mathbf{h}(A)$ is the largest ordinal that is a substructure of A modulo isomorphism (Lemma 2.1.8). Hence $\mathbf{h}(A_\perp) = \alpha$ and $\mathbf{h}(A_\top) = \beta$. □

Corollary 2.5.3 (Height: structural lemma, successor case)**.** *Let A be a* wqo *such that $\mathbf{h}(A) = \alpha + 1$. There exists a partition $A = A_\perp \uplus A_\top$ such that $\mathbf{h}(A_\perp) = \alpha$ and $A_\top$ is* wpo*-isomorphic to an antichain.*

Lemma 2.5.4 (Width: structural lemma)**.** *Let A be a* wqo *such that $\mathbf{w}(A) = \alpha + n$ with $n \geq 1$. There exists A' of width α such that $A' \sqcup \Gamma_n \leq_{\mathsf{st}} A$.*

Proof. If $n = 0$, take $A' = A$.

Otherwise, according to Equation (Res-w), $\mathbf{w}(A) = \sup_{x \in A}(\mathbf{w}(A_{\perp x}) + 1)$, therefore there exists $x_1 \in A$ such that $\mathbf{w}(A_{\perp x_1}) = \alpha + (n-1)$. By induction, we get an antichain $x_1, \ldots, x_n$ such that $\mathbf{w}(A_{\perp x_1, \ldots, x_n}) = \alpha$. Take $A' = A_{\perp x_1, \ldots, x_n}$ and map Γ_n to $x_1, \ldots, x_n$. □

Corollary 2.5.5. *Let A be a* wqo*. If $\mathbf{w}(A) = k < \omega$, then $\Gamma_k \leq_{\mathsf{st}} A$.*

3 - Direct Product

The mot of the direct product of two wqos was first presented as functional in Džamonja et al. (2020), who claimed that $\mathbf{o}(A \cdot B) = \mathbf{o}(A) \cdot \mathbf{o}(B)$. However, as pointed out by Harry Altman (personal correspondance, march 2024), the mot of the direct product is not functional.

Example 1.3.7 (Non-functionality example: Direct product)**.** $D_1 \stackrel{\text{def}}{=} \Gamma_1 + \Gamma_2$ *and* $D_2 \stackrel{\text{def}}{=} \Gamma_2 + \Gamma_1$ *have the same ordinal invariants, but* $\mathbf{o}((\omega + 1) \cdot D_1) \neq \mathbf{o}((\omega + 1) \cdot D_2)$*.*

Proof. Apply Equations (Alc-D-1) and (Alc-D-2) and Table 1.1 to compute the invariants in Table 3.1. □

However, tight bounds are known:

Lemma 3.0.1 (Altman)**.** *Let* A, B *two* wqos*. Then* $\mathbf{o}(A) \cdot \mathbf{o}(B) \leq \mathbf{o}(A \cdot B) \leq \mathbf{o}(A) \odot \mathbf{o}(B)$*.*

These bounds were given to us by Harry Altman, but as we have not found them elsewhere, here is a proof.

Proof. **Proof of the lower bound.** $B \leq_{\mathsf{aug}} \mathbf{o}(B)$ according to Lemma 2.1.8, hence $\mathbf{o}(A \cdot B) \geq \mathbf{o}(A) \cdot \mathbf{o}(B)$ according to Lemma 2.1.5 and Proposition 2.1.9.

Proof of the upper bound, by induction on $\mathbf{o}(B)$. If $\mathbf{o}(B) = \beta + 1$ then there exists $x \in B$ such that $\mathbf{o}(B \setminus \{x\}) = \beta$ and $B \geq_{\mathsf{aug}} B \setminus \{x\} \sqcup \{x\}$ (Lemma 2.5.1). Then $\mathbf{o}(A \cdot B) \leq \mathbf{o}(A \cdot (B \setminus \{x\})) \oplus \mathbf{o}(A) = (\mathbf{o}(A) \odot \beta) \oplus \mathbf{o}(A) = \mathbf{o}(A) \odot (\beta + 1)$ according to Lemma 2.1.5, Proposition 2.1.9, and Table 1.1.

Space	o	h	w
D_1	3	2	2
D_2	3	2	2
$(\omega + 1) \cdot D_1$	$\omega \cdot 3 + 2$	$\omega \cdot 2 + 2$	$\omega \cdot 2 + 1$
$(\omega + 1) \cdot D_2$	$\omega \cdot 3 + 1$	$\omega \cdot 2 + 2$	$\omega \cdot 2 + 1$

Table 3.1: Ordinal invariants of D_1, D_2, through direct product with $\omega+1$

If $\mathbf{o}(B) = \lambda$ limit: For any $x = (a, b) \in A \cdot B$, $(A \cdot B)_{\not\geq x} \leq_{\mathsf{st}} A \cdot B_{\not\geq b}$. Since $\mathbf{o}(B)$ is limit, $\mathbf{o}(B)$ has no maximal elements, hence there exists $b' \geq b$, so $B_{\not\geq b} \leq_{\mathsf{st}} B_{\not\geq b'}$. Therefore $\mathbf{o}(B_{\not\geq s_B}) \leq \mathbf{o}(B_{\not\geq b'}) < \mathbf{o}(B)$. Hence, by induction hypothesis, $\mathbf{o}((A \cdot B)_{\not\geq x}) < \mathbf{o}(A) \odot \mathbf{o}(B)$. We conclude with Equation (Res-o).

□

These two bounds are often equal.

Corollary 3.0.2. *Let A, B two* wqos *such that $\mathbf{o}(B)$ is limit. Then $\mathbf{o}(A \cdot B) = \mathbf{o}(A) \cdot \mathbf{o}(B)$.*

Proof. Apply Lemma A.0.15. □

Actually, we can give an exact formula for $\mathbf{o}(A \cdot B)$.

For any wqo B, let $\mathbf{max\_elt}(B)$ be the number of maximal elements of $B_{/\equiv_B}$. Since the maximal elements of a wqo form an antichain, this invariant is finite.

Lemma 3.0.3 (Maximal elements and mot: structural lemma)**.** *For any* wqo *A, $o(A)$ is of the form $\alpha + m$ for some limit ordinal α (possibly 0) and $\mathbf{max\_elt}(A) \leq m < \omega$. Furthermore, if $\mathbf{max\_elt}(A) = 0$ then $m = 0$.*

Proof. Let $k = \mathbf{max\_elt}(A)$. If $k > 0$ then let $A_\top$ be the set of the maximal elements of A, and $A_\bot \stackrel{\mathsf{def}}{=} A \setminus A_top$. Observe that $A_\top$ is isomorphic to a k-element antichain, hence $\mathbf{o}(A_\top) = k$. Then $A_\bot + A_\top \geq_{\mathsf{aug}} A \geq_{\mathsf{aug}} A_\bot \sqcup A_\top$, therefore $\mathbf{o}(A) = \mathbf{o}(A_\bot) + k$. Thus $\mathbf{o}(A)$ is of the form $\alpha + m$ for some limit ordinal α (possibly 0) and $k \leq m < \omega$.

If $m > 0$ then $\mathbf{o}(A)$ is a successor ordinal $\beta + 1$, hence there exists $x \in A$ such that $\mathbf{o}(A_{\not\geq x} = \beta$ (Equation (Res-o)). If there exists $y >_A x$ then $r(y) \geq \beta + 1 = \mathbf{o}(A)$, contradiction. Hence x is maximal. By contraposition, if $\mathbf{max\_elt}(A) = 0$ then $m = 0$. □

For all $k < \omega$, let pred^k be the predecessor operation iterated k times.

Theorem 3.0.4. *Let B be a* wqo *with $\mathbf{max\_elt}(B) = k$. Then $\mathbf{o}(A \cdot B) = \mathbf{o}(A) \cdot pred^k(\mathbf{o}(B)) + \mathbf{o}(A) \otimes k$.*

Lemma 3.0.5. *Let A, B two* wqos *such that B is finite and $\mathbf{max\_elt}(B) = 1$. Then $\mathbf{o}(A \cdot B) = \mathbf{o}(A) \cdot \mathbf{o}(B)$.*

Proof. We know that B can be written as $B' + \{x\}$. Then $\mathbf{o}(A \cdot B) = \mathbf{o}(A \cdot B') + \mathbf{o}(A) \leq \mathbf{o}(A) \odot \mathbf{o}(B') + \mathbf{o}(A)$. Let $\mathbf{o}(A) = \omega^\alpha \cdot a + \sigma$ with $1 \leq a < \omega$ and $\sigma < \omega^\alpha$. Then $\mathbf{o}(A) \odot \mathbf{o}(B') + \mathbf{o}(A) = \omega^\alpha \cdot a \cdot (\mathbf{o}(B') + 1) + \sigma = \mathbf{o}(A) \cdot \mathbf{o}(B)$. □

Lemma 3.0.6. *Let B be a finite* wpo *with* $\mathbf{max\_elt}(B) = k$. *Then* $\mathbf{o}(A \cdot \mathbf{o}(B)) = \mathbf{o}(A) \cdot (\mathbf{o}(B) - k) + \mathbf{o}(A) \otimes k$.

Proof. B can be partitioned into the antichain of the k maximal elements of B (isomorphic to Γ_k) , and B' the other elements of B. Let $k' = \mathbf{o}(B') = \mathbf{o}(B) - k$.

According to Equation (Alc-D-1) and Table 1.1, $\mathbf{o}(A \cdot \Gamma_k) = \mathbf{o}(A) \otimes k$.

Since $B \leq_{\mathsf{aug}} k' + \Gamma_k$, according to Lemma 2.1.5, Proposition 2.1.9, Equation (Alc-D-2), and Table 1.1 $\mathbf{o}(A \cdot B) \geq \mathbf{o}(A) \cdot k' + \mathbf{o}(A) \otimes k$.

On the other hand, B can be partitioned into k subsets $B_1 \uplus \cdots \uplus B_k$ such that each subset has exactly one maximal element of B as its only maximal elements. Hence B is an augmentation of $B_1 \sqcup \cdots \sqcup B_k$. Let $n_i \stackrel{\mathsf{def}}{=} \mathbf{o}(B_i)$ for $i < k$. Let $\mathbf{o}(A) = \omega^\alpha \cdot a + \sigma$ with $1 \leq a < \omega$ and $\sigma < \omega^\alpha$. Then, according to Lemmas 3.0.5 and 2.1.5 and Table 1.1,

$$\begin{aligned}
\mathbf{o}(A \cdot B) &\leq \bigoplus_{i<k} \mathbf{o}(A) \cdot n_i \text{ (Table 1.1 and Lemma 2.1.5)} \\
&= \bigoplus_{i<k} \omega^\alpha \cdot a \cdot n_i + \sigma \text{ (Equation (A.4))} \\
&= \omega^\alpha \cdot a \cdot \mathbf{o}(B) + \sigma \otimes k \\
&= \omega^\alpha \cdot a \cdot (\mathbf{o}(B) - k) + \omega^\alpha \cdot a \cdot k + \sigma \otimes k \\
&= \omega^\alpha \cdot a \cdot (\mathbf{o}(B) - k) + \sigma + \omega^\alpha \cdot a \cdot k + \sigma \otimes k \text{ (Equation (A.4))} \\
&= \mathbf{o}(A) \cdot (\mathbf{o}(B) - k) + \mathbf{o}(A) \otimes k \; .
\end{aligned}$$

□

Proof of Theorem 3.0.4. B has k maximal elements so $\mathbf{o}(B) = \beta + m$ with β limit and $k \leq m < \omega$. Then according to Lemma 2.5.1, there exists $B_\perp, B_\top$ two wqos such that $\mathbf{o}(B_\perp) = \beta$, and $\mathbf{o}(B_\top) = m$, and $B_\perp + B_\top \geq_{\mathsf{aug}} B \geq_{\mathsf{aug}} B_\perp \sqcup B_\top$. Hence

$$\mathbf{o}(A \cdot B_\perp) + \mathbf{o}(A \cdot B_\top) \geq_{\mathsf{aug}} \mathbf{o}(A \cdot B) \geq_{\mathsf{aug}} \mathbf{o}(A \cdot B_\perp) \oplus \mathbf{o}(A \cdot B_\top) \; .$$

However $\mathbf{o}(A \cdot B_\perp) = \mathbf{o}(A) \cdot \beta$ according to Corollary 3.0.2, and $\mathbf{o}(A \cdot B_\top) = \mathbf{o}(A) \cdot (\mathbf{o}(B) - k) + \mathbf{o}(A) \otimes k$ according to Lemma 3.0.6. We conclude by noticing that $\mathbf{o}(A) \cdot \beta + (\mathbf{o}(A) \cdot (\mathbf{o}(B) - k) + \mathbf{o}(A) \otimes k) = \mathbf{o}(A) \cdot \beta \oplus (\mathbf{o}(A) \cdot (\mathbf{o}(B) - k) + \mathbf{o}(A) \otimes k) = \mathbf{o}(A) \cdot pred^k(\mathbf{o}(B)) + \mathbf{o}(A) \otimes k$. □

This finite invariant can be easily computed for operations of Table 1.1.

Proposition 3.0.7 (Computing **max_elt**)**.** *Let* A, B *be non-empty* wqos*. Then*

- $\mathbf{max\_elt}(A \sqcup B) = \mathbf{max\_elt}(A) + \mathbf{max\_elt}(B)$,
- $\mathbf{max\_elt}(A + B) = \mathbf{max\_elt}(B)$,
- $\mathbf{max\_elt}(A \times B) = \mathbf{max\_elt}(A \cdot B) = \mathbf{max\_elt}(A) \cdot \mathbf{max\_elt}(B)$,
- $\mathbf{max\_elt}(A^*) = \mathbf{max\_elt}(\mathsf{M}^\diamond(A)) = \mathbf{max\_elt}(\mathsf{M}^\mathsf{r}(A)), = \mathbf{max\_elt}(\mathcal{T}(A)) = \mathbf{max\_elt}(\mathsf{P}_\mathsf{f}(A)) = 0$.

4 - Cartesian product

In Section 4.1, we prove an interesting lower bound on the width of the Cartesian product which allows to prove that the width of the Cartesian product is not functional.

In Section 4.2, we compute the width of a Cartesian product of finitely many ordinals, thus generalizing a result from Abraham (1987) which deals with the Cartesian product of two ordinals.

In Section 4.3, we wonder when width reaches mot for the Carteian product, and leverage the result from the previous section to find a sufficient condition.

4.1 . Lower bound on the width of the Cartesian product

Let us recall the notion of transferability defined in Džamonja et al. (2020) to bound the width of the Cartesian product.

Definition 2.4.3. A wqo A is *transferable* if $\mathbf{w}(A_{\not\leq Y}) = \mathbf{w}(A)$ for any finite $Y \subseteq A$.

Theorem 4.1.1 (Theorem 4.16 of Džamonja et al. (2020))**.** *Suppose that A is a transferable* wqo *and β is an ordinal. Then $\mathbf{w}(A \times \beta) \geq \mathbf{w}(A) \cdot \beta$.*

This theorem generalizes easily to:

Corollary 4.1.2. *Let A, B be two wqos. If there exists $A' \leq_{\mathsf{st}} A$ transferable, then $\mathbf{w}(A \times B) \geq \mathbf{w}(A') \cdot \mathbf{o}(B)$.*

This corollary will often be used when $\omega \leq_{\mathsf{st}} A$, to get $\mathbf{w}(A \times B) \geq \mathbf{o}(B)$.

Proof. Observe that $B \geq_{\mathsf{aug}} \mathbf{o}(B)$ (Lemma 2.1.8). Hence $A \times B \geq_{\mathsf{st}} A' \times B \leq_{\mathsf{aug}} A' \times \mathbf{o}(B)$ (Proposition 2.1.9). Hence, according to Theorem 4.1.1 and Lemma 2.1.5, $\mathbf{w}(A \times B) \geq \mathbf{w}(A') \cdot \mathbf{o}(B)$. □

The main result of this section is this:

Theorem 4.1.3 (Bounding the width of the Cartesian product)**.** *Let A, B be two* wqos *such that $\mathbf{w}(A) \geq \omega$. Then $\mathbf{w}(A \times B) \geq \mathbf{w}(A) \cdot \mathbf{o}(B)$.*

Remark 4.1.4. For any partition $A = A_1 \uplus A_2$, $\mathbf{w}(A) \leq \mathbf{w}(A_1) \oplus \mathbf{w}(A_2)$.

Thus if $\mathbf{w}(A) \geq \omega$ is a limit ordinal and A_2 finite, then $\mathbf{w}(A) = \mathbf{w}(A_1)$. In other words, one can remove any finite number of elements of A without changing its width.

Similarly, if $\mathbf{w}(A)$ is additively indecomposable, then either $\mathbf{w}(A_1) = \mathbf{w}(A)$ or $\mathbf{w}(A_2) = \mathbf{w}(A)$ (Lemma A.0.12).

Definition 4.1.5 (n-Transferability)**.** A wqo A is n-transferable iff for all $s \subseteq A$ such that $|s| \leq n$, $\mathbf{w}(A \setminus \downarrow s) = \mathbf{w}(A)$.

Lemma 4.1.6. *Any* wqo *of* mot ω *is transferable.*

Proof. Assume that $\mathbf{o}(A) = \omega$: for all finite $s \subseteq A$, $\mathbf{w}(\downarrow s) \leq \mathbf{o}(\downarrow s) \leq \mathbf{o}(A_{\not\geq s} + |s|) < \omega$. Therefore $\mathbf{w}(A \setminus \downarrow s) = \mathbf{w}(A)$ since $\mathbf{w}(A)$ is limit (see Remark 4.1.4). Hence A is transferable.

□

Lemma 4.1.7. *For all* wqo A *such that* $\mathbf{w}(A)$ *is limit,* A *has a* 1*-transferable substructure of the form* $A_{\not\geq s}$ *for some finite* $s \subseteq A$ *such that* $\mathbf{w}(A_{\not\geq s}) = \mathbf{w}(A)$.

Proof. Since $\mathbf{w}(A)$ is limit, $\mathbf{o}(A) \geq \omega$. By induction on $\mathbf{o}(A)$:

According to Lemma 4.1.6, if $\mathbf{o}(A) = \omega$ then A is transferable, hence 1-transferable.

Assume now that $\mathbf{o}(A) > \omega$. Either A is 1-transferable and $s = \emptyset$ works, or $\mathbf{w}(A \setminus \downarrow x) < \mathbf{w}(A)$ for some $x \in A$. Since $\mathbf{w}(A)$ is limit, $\mathbf{w}(A \setminus \{x\}) = \mathbf{w}(A)$ (Remark 4.1.4). Observe that any antichain of $A \setminus \{x\}$ is either an antichain of $A \setminus \downarrow x$ or $A \setminus \uparrow x$.

Let us label every antichain of $\mathsf{Inco}(A)$ with two color $\perp$ and $\top$ by induction on their rank. Some antichains can be labelled twice. For s of rank 0, label s with $\perp$ if $s \subseteq A \setminus \uparrow x$, and with $\top$ if $s \subseteq A \setminus \downarrow x$. For s of rank $\gamma + 1$, label s with $\perp$ or $\top$ if s has a child of rank γ labelled with $\perp$ or $\top$, respectively.

If s has a limit rank λ then either $\sup_{s' \leq s, s' \text{ labelled } \perp}(rk(s') + 1) = \lambda$ or $\sup_{s' \leq s, s' \text{ labelled } \top}(rk(s') + 1) = \lambda$, not exclusive. Label s with $\perp$ or $\top$ accordingly.

Now every antichain is labelled with one of those two colors, or both. Furthermore, by construction, an antichain labelled with $\perp$ (resp. $\top$) has the same rank in $\mathsf{Inco}(A \setminus \{x\})$ and $\mathsf{Inco}(A \setminus \uparrow x)$ (resp. $\mathsf{Inco}(A \setminus \downarrow x)$). However, we know that the empty antichain has a rank $\mathbf{w}(A \setminus \downarrow x) < \mathbf{w}(A)$ in $\mathsf{Inco}(A \setminus \downarrow x)$. Hence the empty antichain is labelled with $\perp$, and we have $\mathbf{w}(A_{\not\geq x}) = \mathbf{w}(A)$. Moreover, $\mathbf{o}(A_{\not\geq x}) < \mathbf{o}(A)$, hence by induction hypothesis there exists $s \in A_{\not\geq x}$ such that $A_{\not\geq x,s}$ is 1-transferable, and $\mathbf{w}(A_{\not\geq x,s}) = \mathbf{w}(A)$ □

Lemma 4.1.8. *For all* wqo A *such that* $\mathbf{w}(A)$ *is limit,* A *has a transferable substructure of the form* $A_{\not\geq s}$ *for some finite* $s \subseteq A$ *such that* $\mathbf{w}(A_{\not\geq s}) = \mathbf{w}(A)$.

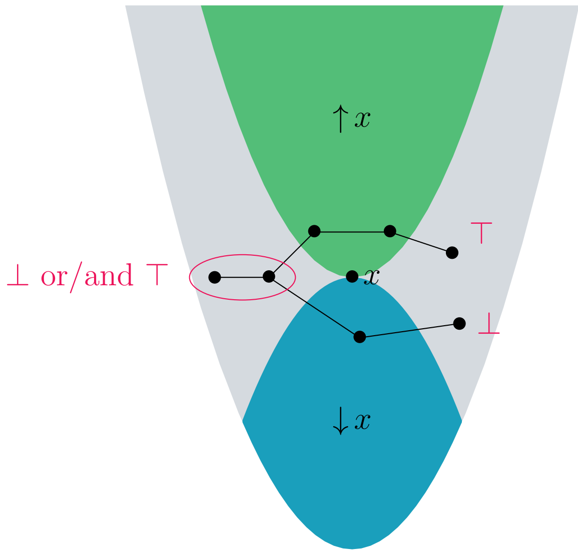


Figure 4.1: Labelling antichains of $\mathsf{Inco}(X \setminus \{x\})$ with $\bot$ and/or $\bot$.

Proof. Since $\mathbf{w}(A)$ is limit, $\mathbf{o}(A) \geq \omega$. By induction on $\mathbf{o}(A)$:

According to Lemma 4.1.6, if $\mathbf{o}(A) = \omega$ then A is transferable, so $s = \emptyset$ works.

Otherwise, according to Lemma 4.1.7, there exists $s \subseteq A$ finite such that $A_{\not\geq s}$ is 1-transferable and $\mathbf{w}(A_{\not\geq s}) = \mathbf{w}(A)$. If $s \neq \emptyset$ then $\mathbf{o}(A_{\not\geq s}) < \mathbf{o}(A)$ hence by induction hypothesis $A_{\not\geq s}$ has a transferable substructure of the form $A_{\not\geq s,s'}$ for some finite $s' \subseteq A_{\not\geq s}$ such that $\mathbf{w}(A_{\not\geq s,s'}) = \mathbf{w}(A)$.

Otherwise $s = \emptyset$ and A is 1-transferable. If A is not transferable then there exists $1 \leq k < \omega$ such that A is k-transferable, but not $(k+1)$-transferable. Then there exists $x_1, \ldots, x_k \in A$ such that $A \setminus \downarrow x_1, \ldots, x_k$ is not 1-transferable, but $\mathbf{w}(A \setminus \downarrow x_1, \ldots, x_k) = \mathbf{w}(A)$. Then according to Lemma 4.1.7, there exists $s \subseteq A \setminus \downarrow x_1, \ldots, x_k, s \neq \emptyset$ such that $\mathbf{w}((A \setminus \downarrow x_1, \ldots, x_k)_{\not\geq s}) = \mathbf{w}(A)$, hence $\mathbf{w}(A_{\not\geq s}) = \mathbf{w}(A)$. Since $\mathbf{o}(A_{\not\geq s}) < \mathbf{o}(A)$ there exists $s' \subseteq A_{\not\geq s}$ such that $\mathbf{w}(A_{\not\geq s,s'}) = \mathbf{w}(A)$ and $A_{\not\geq s,s'}$ is transferable.

□

Proof of Theorem 4.1.3. Let $\mathbf{o}(B) = \beta$. Then $A \times B \leq_{\mathsf{aug}} A \times \beta$ hence $\mathbf{w}(A \times B) \geq \mathbf{w}(A \times \beta)$. W.l.o.g, we can therefore assume that $B = \beta > 0$.

Let $\mathbf{w}(A) = \lambda + k$, with $\lambda \geq \omega$ limit and $0 \leq k < \omega$. Note that $(\lambda + k) \cdot \beta \leq \lambda \cdot \beta + k$ because $\lambda \geq \omega$. Then there exists $s \in \mathsf{Inco}(A)$ such that $|s| = k$ and $\mathbf{w}(A_{\perp s}) = \lambda$. According to Lemma 4.1.8, $A_{\perp s}$ has a transferable substructure A' such that $\mathbf{w}(A') = \lambda$. According to Theorem 4.1.1, $\mathbf{w}(A' \times \beta) \geq \lambda \cdot \beta$. And $A \times B \geq_{\mathsf{st}} \{ (x, 0) \mid x \in s \} \sqcup A' \times B$, hence $\mathbf{w}(A \times B) \geq \mathbf{w}(A) \cdot \beta$.

□

Proof of Example 1.3.5

With Corollary 4.1.2 to give lower bounds and the method of residuals or Lemma 1.2.5 for upper bounds, we can now compute the ordinal invariants of Example 1.3.5. But first we need to treat a simple example; a nice demonstration of how to use augmentations to break a residual into simpler pieces.

We note $A^{\times n} \stackrel{\text{def}}{=} A \times \cdots \times A$ the cartesian product of n copies of a wqo A.

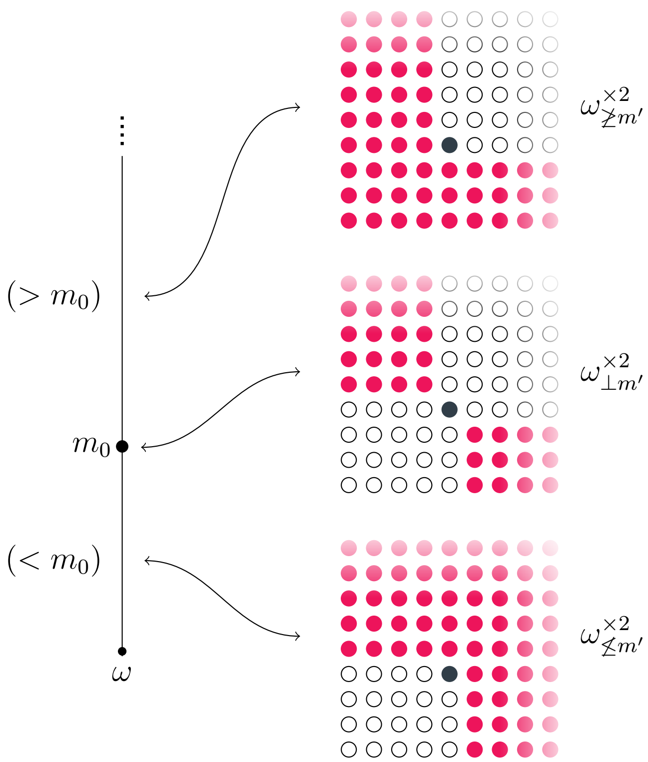


Figure 4.2: Decomposition of the residual of $\mathbb{N}^{\times 3}_{\perp m}$.

Proposition 4.1.9. $\mathbf{w}(\omega^{\times n}) = \omega^{n-1}$ *for* $n \geq 1$.

Proof. Case $n = 1$: $\mathbf{w}(\omega) = 1$.

If $n > 1$, then $\omega^{\times n}$ is a cartesian product of n add. indec. ordinals so it is self-residual hence transferable. Thus according to Corollary 4.1.2 $\mathbf{w}(\omega^{\times n}) \geq \mathbf{w}(\omega) \cdot \mathbf{o}(\omega^{\times (n-1)}) = \omega^{n-1}$.

Let us prove the upper bound by induction on n, initialized in $n = 1$: Assume $\mathbf{w}(\omega^{\times n}) = \omega^{n-1}$ for some n. Let $m = (m_0, \ldots, m_n)$ be any element of $\omega^{\times (n+1)}$, $m' = (m_1, \ldots, m_n)$. We know $(\omega^{\times n})_{<m'}$ is finite, so there exists $k < \omega$ such that $k \geq_{\mathsf{aug}} (\omega^{\times n})_{<m'}$. Then:

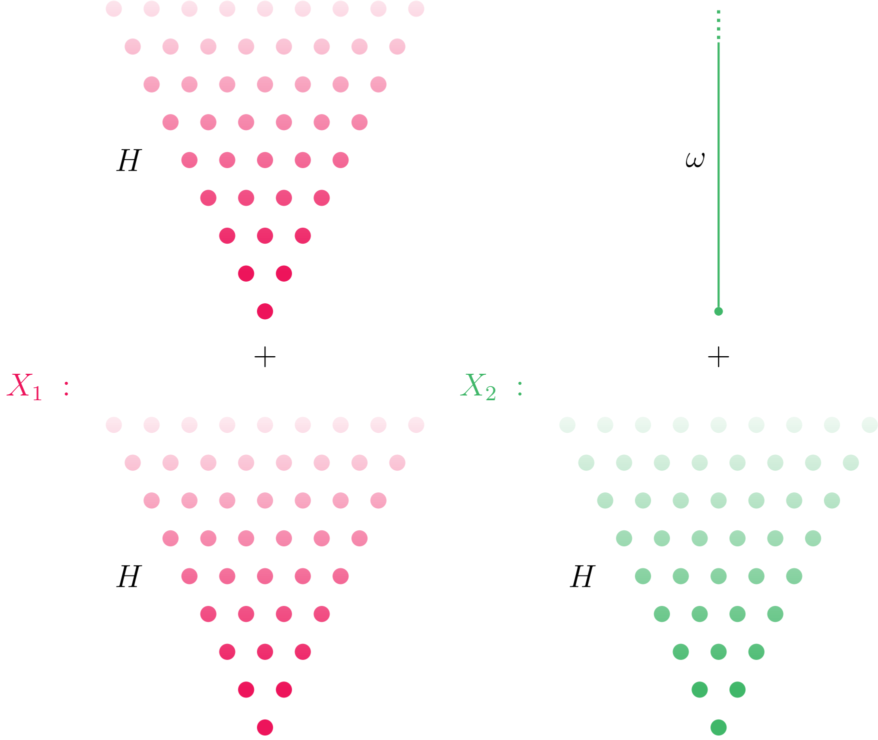


Figure 4.3: X_1 and X_2 of Example 1.3.5 and Example 1.3.6.

$$(\omega^{\times(n+1)})_{\perp m} \geq_{\mathsf{aug}} (< m_0) \times (\omega^{\times n})_{>m'} \sqcup (> m_0) \times (\omega^{\times n})_{<m'} \sqcup \{m_0\} \times (\omega^{\times n})_{\perp m'}$$
$$\geq_{\mathsf{aug}} \Gamma_{m_0} \times \omega^{\times n} \sqcup \omega \times \Gamma_k \sqcup (\omega^{\times n})_{\perp m'} .$$

Therefore by induction hypothesis $\mathbf{w}\left(\omega_{\perp m}^{\times(n+1)}\right) \leq \omega^{n-1} \cdot m_0 \oplus k \oplus \gamma$ with $\gamma < \omega^{n-1}$, hence $\mathbf{w}\left(\omega_{\perp m}^{\times(n+1)}\right) < \omega^n$.

Thus following Equation (Res-w): $\mathbf{w}\left(\omega^{\times(n+1)}\right) = \sup_m \left\{\mathbf{w}(\omega_{\perp m}^{\times(n+1)}) + 1\right\} \leq \omega^n$.

□

Example 1.3.5 (Non-functionality example: Cartesian product)**.** *Let* $X_1 \overset{\mathsf{def}}{=} H + \omega$ *and* $X_2 \overset{\mathsf{def}}{=} H + H$*. Then* X_1 *and* X_2 *have the same ordinal invariants. However* $\mathbf{w}(X_1 \times \omega) \neq \mathbf{w}(X_2 \times \omega)$*.*

Proof. According to Table 1.1, $\mathbf{w}(X_i) = \omega$ and $\mathbf{o}(X_i) = \mathbf{h}(X_i) = \omega \cdot 2$ for $i \in \{1, 2\}$.

According to Proposition 4.1.9, $\mathbf{w}(\omega \times \omega) = \omega$. And $\mathbf{w}(H \times \omega) = \omega^2$, thanks to Lemma 1.2.5 and Theorem 4.1.3.

Space	**o**	**h**	**w**
X_1	$\omega \cdot 2$	$\omega \cdot 2$	ω
X_2	$\omega \cdot 2$	$\omega \cdot 2$	ω
$X_1 \times \omega$	$\omega^2 \cdot 2$	$\omega \cdot 2$	$\omega^2 + \omega$
$X_2 \times \omega$	$\omega^2 \cdot 2$	$\omega \cdot 2$	$\omega^2 \cdot 2$

Observe that $X_1 \geq_{\mathsf{aug}} (H \sqcup \omega)$, thus $\mathbf{w}(X_1 \times \omega) \leq \mathbf{w}(H \times \omega) \oplus \mathbf{w}(\omega \times \omega) = \omega^2 \oplus \omega$ (Lemma 2.1.5). Furthermore, let us show that $(\omega \times \omega, H \times \omega)$ is a family of quasi-incomparable* subsets of $X_1 \times \omega$: For any finite subset Y of $\omega \times \omega$, let $k = \max \{ n \mid (m, n) \in Y \}$. Then $H \times \{ n \in \omega \mid n > k \}$ is isomorphic to $H \times \omega$ and incomparable to Y. Hence according to Lemma 2.3.2, $\mathbf{w}(X_1 \times \omega) = \omega^2 + \omega$. Similarly, $\mathbf{w}(X_2 \times \omega) = \omega^2 \cdot 2 \neq \mathbf{w}(X_1 \times \omega)$.

□

4.2 . Cartesian product of ordinals

Cartesian product of two ordinals

Abraham (1987) used the method of residuals to compute the width of the cartesian product of two ordinals. Extending this result is what motivates this section. Let us recall the main steps of his proof:

Let α, β be two ordinals. According to Equation (Res-w),

$$w(\alpha \times \beta) = \sup_{(x_1, x_2) \in \alpha \times \beta} \left(\mathbf{w}((\alpha \times \beta)_{\perp (x_1, x_2)}) + 1 \right) \ . \tag{4.1}$$

Fix $(x_1, x_2) \in \alpha \times \beta$. Then for any $(y_1, y_2) \in \alpha \times \beta$, $(x_1, x_2) \perp (y_1, y_2)$ if and only if $x_1 < y_1$ and $x_2 > y_2$, or $x_1 > y_1$ and $x_2 < y_2$. In Figure 4.4, the ordinals α and β are represented through vertical lines, and an element (x_1, x_2) of the cartesian product as a segment from x_1 to x_2. Thus, elements incomparable to (x_1, x_2) correspond to segments intersecting (x_1, x_2).

Thus the residual $(\alpha \times \beta)_{\perp (x_1, x_2)}$ is a disjoint union:

$$(\alpha \times \beta)_{\perp (x_1, x_2)} \cong \alpha_{< x_1} \times \beta_{> x_2} \ \sqcup \ \alpha_{> x_1} \times \beta_{< x_2} \ .$$

Observe that $\alpha_{< x_1}$ is isomorphic to x_1, and $\alpha_{> x_1}$ to $\alpha - (x_1 + 1)$. The same reasoning applies to $\beta_{< x_2}$ and $\beta_{> x_2}$. Using Table 1.1 we rewrite Equation (4.1) as:

$$\mathbf{w}(\alpha \times \beta) = \sup_{\substack{x_1 < \alpha \\ x_2 < \beta}} \left(\left(\mathbf{w}(x_1 \times (\beta - x_2)) \oplus \mathbf{w}((\alpha - x_1) \times x_2) \right) + 1 \right) \ .$$

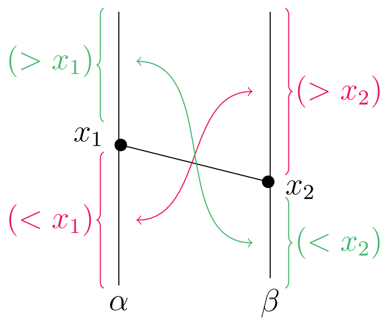


Figure 4.4: Residual of $\alpha \times \beta$ at (x_1, x_2) as a disjoint union.

This equality leads us by induction to the main result of Abraham (1987) (slightly transformed here as to express in one formula what was given as separate results for different cases):

Theorem 4.2.1 (Lemma 3.2 and Theorem 3.4 of Abraham (1987))**.** *For any infinite ordinals $\alpha = \omega^{\alpha_0} \cdot a + \rho$ and $\beta = \omega^{\beta_0} \cdot b + \sigma$, where $\alpha_0, \rho, \beta_0, \sigma$ are ordinals such that $\rho < \omega^{\alpha_0}$ and $\sigma < \omega^{\beta_0}$, and $0 < a, b < \omega$, the width of $\alpha \times \beta$ is computed inductively as:*

$$\mathbf{w}(\alpha \times \beta) = \omega^{\eta} \cdot (a + b - 1) + [\mathbf{w}(\omega^{\alpha_0} \times \sigma) \oplus \mathbf{w}(\omega^{\beta_0} \times \rho)] ,$$

with $\eta = 1 + ((\alpha_0 - 1) \oplus (\beta_0 - 1))$.

To illustrate how computing the width of the product of n ordinals is substantially more complex than the case $n = 2$, let us show why the same proof structure as in Abraham (1987) does not allow us to conclude for the product of $n = 3$ ordinals. Let $X = \alpha_1 \times \alpha_2 \times \alpha_3$ and $x = (x_1, x_2, x_3) \in X$. We can express the residual $X_{\perp x}$ as an union of subsets.

However, unlike the case $n = 2$, this union of disjoint subsets cannot be seen as a disjoint sum of wqos. For instance, observe that the subsets $(> x_1) \times (> x_2) \times (< x_3)$ and $(> x_1) \times (< x_2) \times (< x_3)$ have comparable elements (see Figure 4.5). One can see the residual as an augmentation of a disjoint union, but this only gives us an upper bound on $\mathbf{w}(X)$, without a matching lower bound.

This observation motivated the development of tools to prove refined lower bounds on the width of the cartesian product.

Product of additively indecomposable ordinals

This section computes the width of the product of n additively indecomposable ordinals, for any $2 \leq n < \omega$. Recall the case $n = 2$ from Theorem 4.2.1:

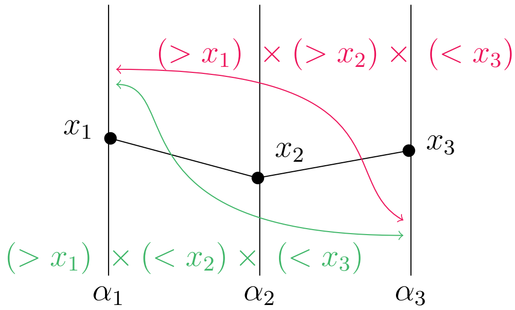


Figure 4.5: Two parts of the residual of $\alpha_1 \times \alpha_2 \times \alpha_3$ at (x_1, x_2, x_3) that have comparable elements.

$\mathbf{w}(\omega^{\alpha_1} \times \omega^{\alpha_2}) = \omega^\eta$ with $\eta = 1 + ((\alpha_1 - 1) \oplus (\alpha_2 - 1))$ for any ordinals $\alpha_1, \alpha_2 > 0$.

Remark 4.2.2. For any ordinals $\alpha \geq \beta$, according to Equations (A.7) and (A.9), $1 + \big((\alpha - 1) \oplus (\beta - 1)\big) = \alpha \oplus (\beta - 1)$.

First let us focus on computing $\mathbf{w}(\omega^{\alpha_1} \times \cdots \times \omega^{\alpha_n})$ when all α_is are infinite, starting with $\mathbf{w}((\omega^\omega)^{\times n})$.

Proposition 4.2.3. $\mathbf{w}((\omega^\omega)^{\times n}) = \omega^{\omega \cdot n}$ *for* $n \geq 2$.

Proof. The case $n = 2$ is an application of Theorem 4.2.1. If $n > 2$:

$$\mathbf{w}((\omega^\omega)^{\times n}) \leq \mathbf{o}((\omega^\omega)^{\times n}) = \omega^{\omega \cdot n} \text{ according to Lemma 1.2.5,}$$
$$\mathbf{w}((\omega^\omega)^{\times n}) \geq \mathbf{w}(\omega^\omega \times \omega^\omega) \cdot \mathbf{o}((\omega^\omega)^{\times (n-2)}) = \omega^{\omega \cdot 2} \cdot \omega^{\omega \cdot (n-2)} = \omega^{\omega \cdot n}$$

according to Theorem 4.1.3 and Table 1.1. □

Lemma 4.2.4. *Let* $X = \omega^{\alpha_1} \times \cdots \times \omega^{\alpha_n}$, *with* $2 \leq n \leq \omega$ *and* $\alpha_1, \ldots, \alpha_n$ *infinite ordinals. Then* $\mathbf{w}(X) = \omega^{\alpha_1 \oplus \cdots \oplus \alpha_n}$.

Proof. We know from Lemma 1.2.5 and Table 1.1 that $\mathbf{w}(X) \leq \mathbf{o}(X) = \omega^{\alpha_1 \oplus \cdots \oplus \alpha_n}$. Observe that $X \leq_{\mathsf{aug}} \omega^{\alpha_1} \times \mathbf{o}(\omega^{\alpha_2} \times \cdots \times \omega^{\alpha_n}) = \omega^{\alpha_1} \times \omega^{\alpha_2 \oplus \cdots \oplus \alpha_n}$. Therefore according to Theorem 4.2.1, $\mathbf{w}(X) \geq \omega^\eta$ with

$$\eta = 1 + ((\alpha_1 - 1) \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1))$$
$$= \alpha_1 \oplus \cdots \oplus \alpha_n \text{ (Equations (A.7) and (A.9)).}$$

□

Theorem 4.2.5. *Let* $X = \omega^{\alpha_1} \times \cdots \times \omega^{\alpha_n}$*, with* $2 \leq n < \omega$ *and* $\alpha_1 \geq \cdots \geq \alpha_n$ *ordinals. Then* $\mathbf{w}(X) = \omega^\eta$*, where* $\eta = 0$ *if* $\alpha_2 = \cdots = \alpha_n = 0$*, otherwise* $\eta = \alpha_1 \oplus \big((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1\big)$.

Proof. Let $k \leq n$ be the integer such that $\alpha_1 \geq \cdots \geq \alpha_k > 0 = \alpha_{k+1} = \cdots = \alpha_n$. If $k \leq 1$, then $X \cong \omega^{\alpha_1}$ so $\mathbf{w}(X) = 1$. Otherwise $k \geq 2$, and $X \cong \omega^{\alpha_1} \times \cdots \times \omega^{\alpha_k}$. Observe that $\alpha_1 \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_k) - 1) = \alpha_1 \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1)$. Hence we assume without loss of generality that $k = n$.

Case $n = 2$ is given by Theorem 4.2.1 and Remark 4.2.2.

If $\alpha_1, \ldots, \alpha_n$ are infinite, then according to Lemma 4.2.4, $\mathbf{w}(X) = \omega^{\alpha_1 \oplus \cdots \oplus \alpha_n} = \omega^{\alpha_1 \oplus \big((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1\big)}$ through Equation (A.7).

If $\alpha_2, \ldots, \alpha_n$ are finite, then according to Theorem 4.1.3

$$\begin{aligned}
\mathbf{w}(X) &\geq \mathbf{w}(\omega^{\alpha_1} \times \omega^{\alpha_2}) \cdot \mathbf{o}(\omega^{\alpha_3} \times \cdots \times \omega^{\alpha_n}) \\
&= \omega^{(\alpha_1 \oplus (\alpha_2 - 1)) + (\alpha_3 \oplus \cdots \oplus \alpha_n)} \text{ (Theorem 4.2.1 and Table 1.1)}, \\
&\overset{A.8}{=} \omega^{\alpha_1 \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1)} .
\end{aligned}$$

Similarly if $\alpha_1, \ldots, \alpha_k$ are infinite and $\alpha_{k+1}, \ldots, \alpha_n$ finite for $2 \leq k < n$, then according to Theorem 4.1.3 and Lemma 4.2.4:

$$\begin{aligned}
\mathbf{w}(X) &\geq \mathbf{w}(\omega^{\alpha_1} \times \cdots \times \omega^{\alpha_k}) \cdot \mathbf{o}(\omega^{\alpha_{k+1}} \times \cdots \times \omega^{\alpha_n}) \\
&= \omega^{\alpha_1 \oplus \cdots \oplus \alpha_n} \\
&= \omega^{\alpha_1 \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1)} .
\end{aligned}$$

Thus $\mathbf{w}(X) \geq \omega^\eta$.

Now we prove the upper bound by induction on $(\alpha_1, \ldots, \alpha_n)$ with the cartesian product ordering:

The induction is initialized with $\mathbf{w}(\omega^{\times n}) = \omega^{n-1}$ from Proposition 4.1.9.

Let $x = (x_1, \ldots, x_n)$ be an element of X. For all $i \in [1, n]$ there exists $0 \leq \alpha'_i < \alpha_i$ and $m_i \in \omega$ such that $x_i \leq \omega^{\alpha'_i} \cdot m_i < \omega^{\alpha_i}$. The residual $X_{\perp x}$ is a substructure of an augmentation of a disjoint sum of terms of the form $\left(\bigtimes_{i \in I} (< x_i)\right) \times \left(\bigtimes_{i \notin I} (\geq x_i)\right)$ with $I \subsetneq [1, n], I \neq \emptyset$. Observe that:

$$\begin{aligned}
\left(\bigtimes_{i \in I} (< x_i)\right) \times \left(\bigtimes_{i \notin I} (\geq x_i)\right) &\leq_{\mathsf{st}} \left(\bigtimes_{i \in I} \omega^{\alpha'_i} \cdot m_i\right) \times \left(\bigtimes_{i \notin I} \omega^{\alpha_i}\right) \\
&\geq_{\mathsf{aug}} \left(\bigtimes_{i \in I} \omega^{\alpha'_i} \times \Gamma_{m_i}\right) \times \left(\bigtimes_{i \notin I} \omega^{\alpha_i}\right) ,
\end{aligned}$$

therefore

$$\mathbf{w}(X_{\perp x}) \leq \bigoplus_{I \subsetneq [1,n], I \neq \emptyset} \mathbf{w}\left(\left(\bigtimes_{i \in I} \omega^{\alpha'_i}\right) \times \left(\bigtimes_{i \notin I} \omega^{\alpha_i}\right)\right) \cdot \prod_{i \in I} m_i .$$

By induction hypothesis, $\mathbf{w}\left(\left(\underset{i\in I}{\times}\omega^{\alpha_i'}\right)\times\left(\underset{i\notin I}{\times}\omega^{\alpha_i}\right)\right) = \omega^{\eta'}$, with $\eta' < \alpha_1 \oplus (\alpha_2 \oplus \cdots \oplus \alpha_n - 1)$ according to Remark 4.2.6.

Therefore $\mathbf{w}(X) \leq \omega^{\alpha_1 \oplus (\alpha_2 \oplus \cdots \oplus \alpha_n - 1)}$ through Equation (Res-w). □

Remark 4.2.6. Since the natural sum and the left subtraction are strictly increasing in both arguments or on the left argument, respectively, then for any ordinals $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ with $2 \leq n < \omega$, if $(\alpha_1, \ldots, \alpha_n) < (\beta_1, \ldots, \beta_n)$ component-wise then $\alpha_1 \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1) < \beta_1 \oplus ((\beta_2 \oplus \cdots \oplus \beta_n) - 1)$.

Corollary 4.2.7. *If α_1 and α_2 are infinite, then $\alpha_1 \oplus ((\alpha_2 \oplus \cdots \oplus \alpha_n) - 1) = \alpha_1 \oplus \cdots \oplus \alpha_n$, hence $\mathbf{w}(X) = \mathbf{o}(X)$ according to Table 1.1. Otherwise $\mathbf{w}(X) < \mathbf{o}(X)$.*

Product of infinite ordinals

This section leverage our result on the width of the product of additively indecomposable ordinals (Theorem 4.2.5) to compute the width of the product of infinite ordinals (Theorem 4.2.8).

Let $X = \alpha_1 \times \cdots \times \alpha_n$ be a cartesian product of n infinite ordinals $(\alpha_i)_{1 \leq i \leq n}$. For any $i \in [1, n]$, α_i is written as $\sum_{j<l_i} \omega^{\alpha_{i,j}}$ in CNF, i.e. $\alpha_{i,0} \geq \cdots \geq \alpha_{i,l_i-1}$.

We partition X into disjoint subsets we call slices: let $Sl(X) \overset{\text{def}}{=} l_1 \times \cdots \times l_n$ be the set of slice indices. For any $s = (s(1), \ldots, s(n)) \in Sl(X)$, we define the slice X_s as

$$X_s \overset{\text{def}}{=} \underset{i\in[1,n]}{\times} X_{s,i}$$

where $X_{s,i}$ is the interval of α_i whose elements are bigger than or equal to $\sum_{j \leq s(i)-1} \omega^{\alpha_{i,j}}$ (or 0 if $s(i) = 0$) and strictly smaller than $\sum_{j \leq s(i)} \omega^{\alpha_{i,j}}$.

Observe that X_s is isomorphic to $\underset{i\in[1,n]}{\times} \omega^{\alpha_{i,s(i)}}$. Therefore we know $\mathbf{w}(X_s)$ through Theorem 4.2.5.

We say $s \in Sl(X)$ is grounded if there exists $k \in [1, n]$ such that $s(k) = 0$. Let $Gr(X) \overset{\text{def}}{=} \{s \in Sl(X) \mid \exists k \in [1, n], s(k) = 0\}$ the set of grounded slice indices. We denote the cardinal of $Gr(X)$ with $L = \prod l_i - \prod (l_i - 1)$.

Theorem 4.2.8. *Let $X = \alpha_1 \times \cdots \times \alpha_n$ be a cartesian product of n infinite ordinals. Then*

$$\mathbf{w}(X) = \bigoplus_{s\in Gr(X)} \mathbf{w}\big(\underset{i\in[1,n]}{\times} \omega^{\alpha_{i,s(i)}}\big)\ . \tag{4.2}$$

We will first prove the upper bound $\mathbf{w}(X) \leq \bigoplus_{s\in Gr(X)} \mathbf{w}(X_s)$, then the lower bound $\mathbf{w}(X) \geq \bigoplus_{s\in Gr(X)} \mathbf{w}(X_s)$.

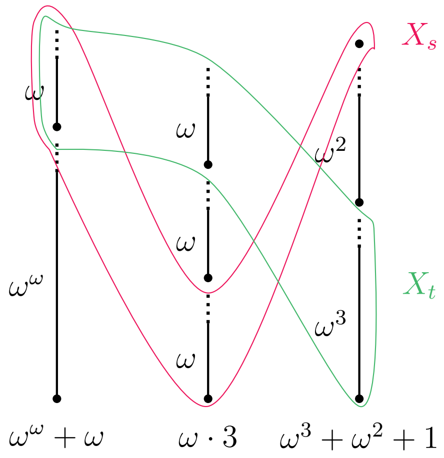


Figure 4.6: Slices and grounded slices: $X = (\omega^\omega+\omega)\times(\omega\cdot 3)\times(\omega^3+\omega^2+1)$, X_s and X_t for $s = (1, 0, 2)$, $t = (1, 2, 0)$ are both grounded.

Proof of the upper bound of Theorem 4.2.8.

For any slices $s, t \in Sl(X)$, we write $s \prec t$ if and only if for all $i \in [1, n]$, $s(i) < t(i)$. Observe that, for any s, t such that $s \prec t$:

- for any $x \in X_s$, $x' \in X_t$, we have $x <_X x'$.
- for any $i \in [1, n]$, $\alpha_{i,s(i)} \geq \alpha_{i,t(i)}$, therefore $\mathbf{w}(X_s) \geq \mathbf{w}(X_t)$ according to Theorem 4.2.5 and Remark 4.2.6.

We define a surjective function $g : Sl(X) \to Gr(X)$ which maps any slice index to a grounded slice index:

$$g(s)(i) \overset{\mathsf{def}}{=} s(i) - k \text{ with } k = \min_{i\in[1,n]} s(i) \ .$$

This surjection has interesting properties:

If s is grounded then $g(s) = s$, otherwise $g(s) \prec s$. Thus $\mathbf{w}(X_s) \leq \mathbf{w}(X_{g(s)})$.

For any distinct $s, t \in Sl(X)$, such that $g(s) = g(t)$, $s \prec t$ or $s \succ t$.

Thus X is an augmentation of the disjoint sum of direct sums of slices grouped by image through g, as illustrated in Figure 4.7:

$$X \geq_{\mathsf{aug}} \bigsqcup_{s\in Gr(X)} \sum_{s'\in g^{-1}(s)} X_{s'} \ .$$

Therefore, according to Table 1.1,

$$\mathbf{w}(X) \leq \bigoplus_{s\in Gr(X)} \max_{s'\in g^{-1}(s)} \mathbf{w}(X_{s'}) = \bigoplus_{s\in Gr(X)} \mathbf{w}(X_s) \ .$$

□

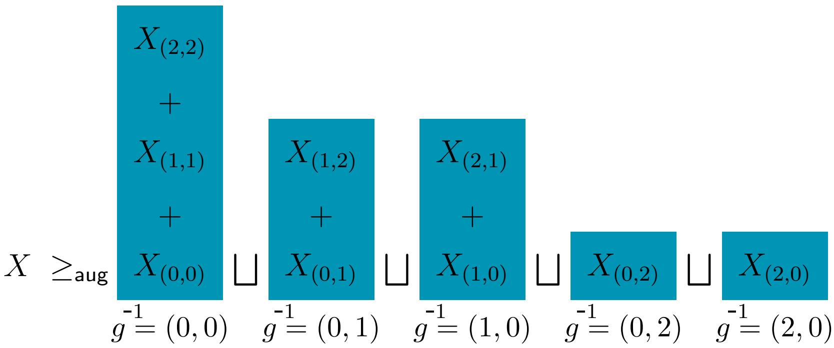


Figure 4.7: Relations between slices when $Sl(X) = 3 \times 3$.

We need to introduce a few notations before proving the lower bound of Theorem 4.2.8.

For any finite subset Y of X, we define $\xi(Y,i)$ as the maximum of the i-th components of elements of Y which are less than $\omega^{\alpha_{i,0}}$:

$$\xi(Y,i) \stackrel{\mathsf{def}}{=} \max \left\{ y(i) + 1 \mid y \in Y, y(i) < \omega^{\alpha_{i,0}} \right\} \cup \{0\} \ .$$

We define a function trim which given a slice X_s and a finite subset of X outputs a subset of X_s:

$$\operatorname{trim}(X_s, Y) \stackrel{\mathsf{def}}{=} \bigtimes_{i \in [1,n]} \operatorname{trim}(X_{s,i}, Y) \ ,$$

where

$$\operatorname{trim}(X_{s,i}, Y) \stackrel{\mathsf{def}}{=} \begin{cases} \left\{ \delta \in X_{s,i} \mid \xi(Y,i) \leq \delta \right\} & \text{if } s(i) = 0, \\ X_{s,i} & \text{otherwise.} \end{cases}$$

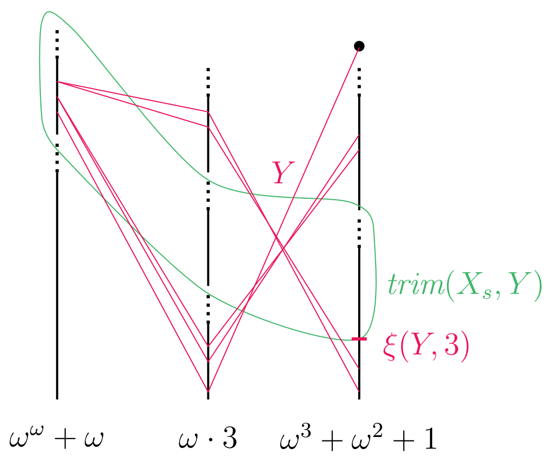


Figure 4.8: $trim(X_s, Y)$, for $s = (1,1,0)$, is incomparable to Y

Lemma 4.2.9. *For all finite set $Y \in X$, and all slice index $s \in Sl(X)$, $trim(X_s, Y)$ is isomorphic to X_s.*

Proof. For all $i \in [1, n]$, $trim(X_{s,i}, Y)$ is isomorphic to $X_{s,i}$: If $s(i) > 0$ then $trim(X_{s,i}, Y) = X_{s,i}$. Otherwise $s(i) = 0$ and $trim(X_{s,i}, Y) = \{\delta \in X_{s,i} \mid \xi(Y, i) \leq \delta\} \cong \omega^{\alpha_{i,0}} - \xi(Y, i)$. Since α_i is infinite, $\omega^{\alpha_{i,0}}$ is infinite indecomposable, and $\xi(Y, i) < \omega^{\alpha_{i,0}}$ since Y is finite and $\omega^{\alpha_{i,0}}$ limit. Therefore $\omega^{\alpha_{i,0}} - \xi(Y, i) = \omega^{\alpha_{i,0}} \cong X_{s,i}$. □

Since $(Gr(X), \leq_\times)$ ordered component-wise is a finite partial order, any enumeration $s_1, \ldots, s_L \in Gr(X)$ is a linearisation of $Gr(X)$ iff for all $i < j \in [1, L]$, $s_i \not\leq s_j$.

Lemma 4.2.10. *For any such linearisation, $(X_{s_i})_{1 \leq i \leq L}$ is a family of quasi-incomparable* subsets of X.*

Proof. Fix $k \in [2, L]$ and Y a finite subset of $X_{s_1} \cup \cdots \cup X_{s_{k-1}}$. Then we define $X'_{s_k} \subseteq X_{s_k}$ as $trim(X_{s_k}, Y)$. According to Lemma 4.2.9, X'_{s_k} is isomorphic to X_{s_k}.

Let us show $X'_{s_k} \perp Y$: We pick $x \in X'_{s_k}, y \in Y$ two elements of X. There exists $j < k$ such that $y \in X_{s_j}$. Since $s_1, \ldots, s_L$ is a linearisation of $Gr(X)$, $s_j \not\leq s_k$. Therefore there exists $i_1 \in [1, n]$ such that $s_j(i_1) > s_k(i_1)$, hence $y(i_1) > x(i_1)$. Since s_j is grounded, there exists $i_2 \in [1, n]$ such that $s_j(i_2) = 0$. If $s_k(i_2) > 0$ then $x(i_2) \geq \omega^{\alpha_{i_1,0}} > y(i_2)$. otherwise $x(i_2) \geq \xi(Y, i_2) > y(i_2)$. Therefore $x \perp y$.

□

Lemma 4.2.11. *There exists a linearisation $s_1, \ldots, s_L$ of $(Gr(X), \leq_\times)$ such that $\mathbf{w}(X_{s_L}) + \cdots + \mathbf{w}(X_{s_1}) = \bigoplus_{s \in Gr(X)} \mathbf{w}(X_s)$.*

Proof. According to Theorem 4.2.5, for all $s \in Sl(X)$, $\mathbf{w}(X_s)$ can be written under the form $\mathbf{w}(X_s) = \omega^{\eta_s}$ for some ordinal η_s. Observe that for all distinct grounded slices s, s', $\eta_s < \eta_{s'}$ implies that $s_i \not\leq s_j$ according to Remark 4.2.6. Therefore there exists a linearisation $s_1, \ldots, s_L$ such that $\eta_{s_1} \leq \cdots \leq \eta_{s_L}$, which means that $\mathbf{w}(X_{s_L}) + \cdots + \mathbf{w}(X_{s_1}) = \bigoplus_{s \in Gr(X)} \mathbf{w}(X_s)$.

□

Proof of the lower bound of Theorem 4.2.8.
Let $s_1, \ldots, s_L$ be a linearisation of $(Gr(X), \leq_\times)$ such that $\mathbf{w}(X_{s_L}) + \cdots + \mathbf{w}(X_{s_1}) = \bigoplus_{s \in Gr(X)} \mathbf{w}(X_s)$. Such an ordering exists according to Lemma 4.2.11. According to Lemma 4.2.10, $(X_{s_j})_{j \in [1,L]}$ is a family of quasi-incomparable* subsets, hence according to Lemma 2.3.2,

$$\mathbf{w}(X) \;\geq\; \mathbf{w}(X_{s_L}) + \cdots + \mathbf{w}(X_{s_1}) \;=\; \bigoplus_{s \in Gr(X)} \mathbf{w}(X_s)\;.$$

□

We can rewrite Equation (4.2) from Theorem 4.2.8 in a way that makes our result easier to compare with Theorem 4.2.1.

Corollary 4.2.12 (Inductive expression of $\mathbf{w}(X)$)**.** *Let $X = \alpha_1 \times \cdots \times \alpha_n$ be a cartesian product of n infinite ordinals. For $i \in [1,n]$, let $\alpha_i = \omega^{\alpha_{i,0}} \cdot a_i + \sigma_i$ where $\alpha_{i,0}$ and σ_i are ordinals such that $\sigma_i < \omega^{\alpha_{i,0}}$, and $0 < a_i < \omega$. Then*

$$\mathbf{w}(X) = \bigoplus_{\emptyset \neq I \subseteq [1,n]} \mathbf{w}(X_I) \otimes \left(\prod_{i \in I} a_i - \prod_{i \in I} (a_i - 1) \right) ,$$

where $X_I \stackrel{\mathsf{def}}{=} \left(\bigtimes_{i \in I} \omega^{\alpha_{i,0}} \right) \times \left(\bigtimes_{i \notin I} \sigma_i \right)$ *for all* $I \subseteq [1,n], I \neq \emptyset$.

As expected, when $n = 2$ this is exactly Abraham's formula (Theorem 4.2.1).

Proof. For all $i \in [1,n]$, each α_i can be written uniquely as $\sum_{j < l'_i} \omega^{\alpha'_{i,j}} \cdot a_{i,j}$ with $\alpha'_{i,0} > \cdots > \alpha'_{i,l'_i - 1}$. Let $Sl'(X) \stackrel{\mathsf{def}}{=} l'_1 \times \cdots \times l'_n$ and let $Gr'(X)$ be the grounded slices of $Sl'(X)$. There exists a unique function $pack : Sl(X) \to Sl'(X)$ such that $\alpha_{i,s(i)} = \alpha'_{i,pack(s)(i)}$ for all $s \in Sl(X)$, $i \in [1,n]$. Observe that for all $s \in Sl'(X)$, $|pack^{-1}(s)| = \prod_{i \in [1,n]} a_{i,s(i)}$.

For all $s \in Gr'(X)$ let

$$\begin{aligned} k_s &\stackrel{\mathsf{def}}{=} |pack^{-1}(s) \cap Gr(X)| \\ &= \left(\prod_{s(i)=0} a_{i,0} - \prod_{s(i)=0} (a_{i,0} - 1) \right) \cdot \prod_{s(i)>0} a_{i,s(i)} . \end{aligned}$$

Let $X'_s = \times_{i \in [1,n]} \omega^{\alpha'_{i,s(i)}}$ for all $s \in Sl'(X)$. Observe that for all $s \in Sl(X)$, $X'_{pack(s)} \cong X_s$. Then

$$\bigoplus_{s \in Gr(X)} \mathbf{w}(X_s) = \bigoplus_{s \in Gr'(X)} \mathbf{w}(X'_s) \otimes k_s ,$$

Now let us compute $\mathbf{w}(X_I)$ for some $I \in [1,n], I \neq \emptyset$. By definition, X_I is a Cartesian product of ordinals whose slices are all grounded. X_I can also be seen as a substructure of X:

$$X_I \cong X \cap \left(\bigcup_{s \in Gr(X), s(i)=0 \text{ iff } i \in I} X_s \right) .$$

Hence according to Theorem 4.2.8,

$$\begin{aligned} \mathbf{w}(X_I) &= \bigoplus_{s \in Gr(X), s(i)=0 \text{ iff } i \in I} \mathbf{w}(X_s) \\ &= \bigoplus_{s \in Gr'(X), s(i)=0 \text{ iff } i \in I} \mathbf{w}(X'_s) \otimes \prod_{i \notin I} a_{i,s(i)} , \end{aligned}$$

since $|pack^{-1}(s) \cap \{t \in Gr(X), t(i) = 0 \text{ iff } i \in I\}| = \prod_{i \notin I} a_{i,s(i)}$.

Furthermore, $(\{s \in Gr(X), s(i) = 0 \text{ iff } i \in I\})_{I \subseteq [1,n], I \neq \emptyset}$ is a partition of $Gr(X)$. Therefore

$$\mathbf{w}(X) = \bigoplus_{\emptyset \neq I \subseteq [1,n]} \mathbf{w}(X_I) \otimes \left(\prod_{i \in I} a_i - \prod_{i \in I} (a_i - 1) \right) \text{ as claimed.}$$

□

Product of ordinals where at least one is infinite

This section extends our result on the width of the product of infinite ordinals (Theorem 4.2.8) to the width of the product of finite and infinite ordinals (Theorem 4.2.16).

Lemma 4.2.13. *Let A be a wqo, and $n < \omega$. Then $\mathbf{w}(A \times \Gamma_n) = \mathbf{w}(A) \otimes n$.*

Proof. Observe that $A \times \Gamma_n$ is the disjoint sum of n copies of A. Therefore $\mathbf{w}(A \times \Gamma_n) = \mathbf{w}(A) \oplus \cdots \oplus \mathbf{w}(A) = \mathbf{w}(A) \otimes n$. □

Lemma 4.2.14. *For X a cartesian product of finitely many additively indecomposable ordinals, and $k < \omega$, $\mathbf{w}(X \times k) = \mathbf{w}(X) \otimes k$*

Proof. Since $X \times k \geq_{\mathsf{aug}} X \times \Gamma_k$, according to Lemma 4.2.13 $\mathbf{w}(X \times k) \leq \mathbf{w}(X) \otimes k$. On the other hand, $\mathbf{w}(X) \geq \omega$ hence $\mathbf{w}(X \times k) \geq \mathbf{w}(X) \cdot k$ with Theorem 4.1.3. According to Theorem 4.2.5 $\mathbf{w}(X)$ is add. indec., therefore $\mathbf{w}(X) \cdot k \overset{A.0.11}{=} \mathbf{w}(X) \otimes k$.

□

Lemma 4.2.15. *Let X be a cartesian product of finitely many infinite ordinals, and $k < \omega$. There exists a linearisation $s_1, \ldots, s_L$ of $(Gr(X), \leq_\times)$ such that $(X_{s_i} \times k)_{i \in [1,L]}$ is a family of quasi-incomparable* subsets of $X \times k$, and $\mathbf{w}(X_{s_L} \times k) + \cdots + \mathbf{w}(X_{s_1} \times k) = \mathbf{w}(X) \otimes k$.*

Proof. According to Lemma 4.2.11, there exists a linearisation $s_1, \ldots, s_L$ of $(Gr(X), \leq_\times)$ such that $\mathbf{w}(X_{s_L}) + \cdots + \mathbf{w}(X_{s_1}) = \bigoplus_{s \in Gr(X)} \mathbf{w}(X_s)$. We claim that $(Z_i)_{i \in [1,L]} \overset{\mathsf{def}}{=} (X_{s_i} \times k)_{i \in [1,L]}$ is a family of quasi-incomparable* subsets of $X \times k$: For all $i \in [2, L]$, for all finite $Y \in Z_1 \cup \cdots \cup Z_{i-1}$, we define Y' as the projection of Y on X. According to Lemma 4.2.10, $(X_{s_i})_{i \in [1,L]}$ is a family of quasi-incomparable* subsets, therefore there exists $X'_{s_i} \subseteq X_{s_i}$ such that $X'_{s_i} \perp Y'$ and $X'_{s_i} \cong X_{s_i}$. Let $Z'_i = X'_{s_i} \times k \subseteq Z_i$. Then $Z'_i \perp Y$ and $Z'_i \cong Z_i$.

For all $s \in Gr(X)$, $\mathbf{w}(X_s \times k) = \mathbf{w}(X_s) \otimes k$ according to Lemma 4.2.14. Thus,

$$\mathbf{w}(Z_L) + \cdots + \mathbf{w}(Z_1) = \left(\bigoplus_{s \in Gr(X)} \mathbf{w}(X_s) \right) \otimes k = \mathbf{w}(X) \otimes k \ ,$$

according to Theorem 4.2.8. □

Theorem 4.2.16. *For X an infinite ordinal or a cartesian product of finitely many infinite ordinals, and $k_1, \dots, k_m < \omega$,*

$$\mathbf{w}(X \times k_1 \times \cdots \times k_m) = \mathbf{w}(X) \otimes \prod_{i \leq n} k_i$$

Proof. Let $k = \prod_{i \leq n} k_i$. We know that $X \times k_1 \times \cdots \times k_m \geq_{\mathsf{aug}} X \times \Gamma_k$ so by Lemma 4.2.13 we know that $\mathbf{w}(X \times k_1 \times \cdots \times k_m) \leq \mathbf{w}(X) \otimes k$.

If X is an infinite ordinal, then $X \geq_{\mathsf{st}} \omega \times k_1 \times \cdots \times k_n$ and ω is transferable, hence according to Corollary 4.1.2 $\mathbf{w}(X) \geq \mathbf{w}(\omega) \cdot \mathbf{o}(k_1 \times \cdots \times k_n) = k$.

If X is a cartesian product of infinite ordinals, then observe that $X \times k_1 \times \cdots \times k_m \leq_{\mathsf{aug}} X \times (\prod_{i \leq n} k_i)$. Therefore $\mathbf{w}(X \times k_1 \times \cdots \times k_m) \geq \mathbf{w}(X \times (\prod_{i \leq n} k_i)) \geq \mathbf{w}(X) \otimes (\prod_{i \leq n} k_i)$ according to Lemmas 2.3.2 and 4.2.15.

□

Product of finite ordinals

The case of the cartesian product of finite ordinals is a finite poset, thus its width coincides with the length of its largest antichain. For the sake of completeness, we recall a classical result that characterizes its width.

Let $k_1, \dots, k_n > 0$ be n finite ordinals, and $p_1, \dots, p_n$ some distinct prime numbers. Observe that $X \stackrel{\mathsf{def}}{=} k_1 \times \cdots \times k_n$ is isomorphic to the poset of the divisors of $p_1^{k_1-1} \cdots p_n^{k_n-1}$ ordered by divisibility. Therefore, according to Theorem 1 of de Bruijn et al. (1951):

Theorem 4.2.17. *Let $X \stackrel{\mathsf{def}}{=} k_1 \times \cdots \times k_n$ be a cartesian product of finite ordinals. Then $\mathbf{w}(X) = |Y|$, with*

$$Y = \left\{ (m_1, \dots, m_n) \in X \;\middle|\; \sum m_i = \left\lfloor \frac{1}{2} \sum (k_i - 1) \right\rfloor \right\}$$

a maximal antichain of X.

For instance,

$$\mathbf{w}(2^{\times n}) = \left| \left\{ (m_1, \dots, m_n) \in \{0,1\}^{\times n} \;\middle|\; \sum m_i = \left\lfloor \frac{n}{2} \right\rfloor \right\} \right|$$
$$= \binom{n}{\lfloor \frac{n}{2} \rfloor} \text{ the } n\text{th central binomial coefficient.}$$

Similarly, $\mathbf{w}(3^{\times n})$ is equal to the central trinomial coefficient, defined as the largest coefficient of the polynomial $(1 + x + x^2)^n$. We can compute it efficiently:

$$\mathbf{w}(3^{\times n}) = \sum_{0 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{i} \binom{n-i}{i}.$$

This leads to a slightly different characterization of $\mathbf{w}(X)$ which can be deduced from Theorem 4.2.17:

Corollary 4.2.18. *For $X = k_1 \times \cdots \times k_n$ a cartesian product of finite ordinals, $\mathbf{w}(X)$ is equal to the central coefficient of the polynomial $P_{k_1} \times \cdots \times P_{k_n}$, where $P_{k_i}(x) = 1 + x + \cdots + x^{k_i - 1}$.*

Proof. The central coefficient of $P_{k_1} \times \cdots \times P_{k_n}$ is the coefficient of $x^{\lfloor (\sum k_i - 1)/2 \rfloor}$. □

Summing up Let us gather Theorems 4.2.5, 4.2.8, 4.2.16 and 4.2.17 into one result:

Theorem 4.2.19. *Let $X = \alpha_1 \times \cdots \times \alpha_n$, with $2 \leq n < \omega$ and $\alpha_1 \geq \cdots \geq \alpha_n$ ordinals. Let α_i be written as $\sum_{j<l_i} \omega^{\alpha_{i,j}}$ in CNF, i.e., $\alpha_{i,0} \geq \cdots \geq \alpha_{i,l_i-1}$, for all $1 \leq i \leq n$.*

Let k be the smallest i such that $\alpha_{i,0} = 0$.

Then

$$\mathbf{w}(X) = \begin{cases} \left|\left\{(m_1, \ldots, m_n) \in X \;\middle|\; \sum m_i = \left\lfloor \frac{1}{2} \sum (\alpha_i - 1) \right\rfloor \right\}\right| & \text{if } k = 1, \\ \bigoplus\limits_{\substack{s \in l_1 \times \cdots \times l_{k-1}, \\ \min s = 0}} \omega^{\eta\left(\alpha_{1,s(1)}, \ldots, \alpha_{k-1,s(k-1)}\right)} \otimes \left(\prod_{k \leq i \leq n} \alpha_i\right) & \text{otherwise,} \end{cases}$$

where, for any $\beta_1 \geq \ldots, \beta_m$, $\eta(\beta_1, \ldots, \beta_m) = 0$ if $\beta_2 = \cdots = \beta_m = 0$, otherwise $\eta(\beta_1, \ldots, \beta_m) = \beta_1 \oplus \big((\beta_2 \oplus \cdots \oplus \beta_m) - 1\big)$.

4.3 . Application to the cartesian product of wqos

We computed the width of the cartesian product of any number of ordinals. Let us now demonstrate how this result can be extended to the cartesian product of more complex wqos.

In view of $\mathbf{w}((\omega^\omega)^{\times n}) = \mathbf{o}((\omega^\omega)^{\times n})$ (Proposition 4.2.3), one wonders if more generally $\mathbf{w}(X)$ reaches $\mathbf{o}(X)$ when X is a cartesian product of ordinals, for instance when the ordinals are large enough? It turns out that we can exactly characterize the cartesian products of ordinals such that width and m.o.t. coincide:

Theorem 4.3.1. *Let $Z = \alpha_1 \times \cdots \times \alpha_n \times k_1 \times \cdots \times k_m$ with $n, m < \omega$ and $n > 0$, such that $\alpha_1, \ldots, \alpha_n$ are infinite ordinals, and $0 < k_1, \ldots, k_m < \omega$. Now $\mathbf{w}(Z) = \mathbf{o}(Z)$ if and only if there exist:*

- *$i \in [1, n]$ such that α_i is additively indecomposable, and*
- *$j_1 \neq j_2 \in [1, n]$ such that the Cantor normal forms of α_{j_1} and α_{j_2} only have infinite exponents (i.e., can be written as $\omega^\omega \cdot \beta$ with β some ordinal).*

Note that i can be equal to j_1 or j_2.

Proof. Let $X = \alpha_1 \times \cdots \times \alpha_n$ with α_i written $\sum_{j<l_i} \omega^{\alpha_{i,j}}$ in CNF for all $i \in [1, n]$. We will reuse the notations $Sl(X)$ and $Gr(X)$.

According to Theorem 4.2.16, $\mathbf{w}(Z) = \mathbf{w}(X) \otimes k_1 \otimes \cdots \otimes k_m$, and $\mathbf{o}(Z) = \mathbf{o}(X) \otimes k_1 \otimes \cdots \otimes k_m$ according to Table 1.1. Therefore $\mathbf{w}(Z) = \mathbf{o}(Z)$ if and only if $\mathbf{w}(X) = \mathbf{o}(X)$.

We express $\mathbf{o}(X)$ in a form that allow us to compare it easily to $\mathbf{w}(X)$:

$$\begin{aligned}
\mathbf{o}(X) &= \bigotimes_{i\in[1,n]} \alpha_i \text{ according to Table 1.1} \\
&= \bigoplus_{s\in Sl(X)} \left(\bigotimes_{i\in[1,n]} \omega^{\alpha_{i,s(i)}} \right) \text{ by distributivity} \\
&= \bigoplus_{s\in Sl(X)} \mathbf{o}(X_s) ,
\end{aligned}$$

and

$$\mathbf{w}(X) = \bigoplus_{s\in Gr(X)} \mathbf{w}(X_s) \text{ according to Theorem 4.2.8.}$$

According to Lemma 1.2.5, for every slice $s \in Sl(X)$, $0 < \mathbf{w}(X_s) \leq \mathbf{o}(X_s)$. Moreover $Gr(X) \subseteq Sl(X)$. Therefore $\mathbf{w}(X) = \mathbf{o}(X)$ if and only if $Gr(X) = Sl(X)$ and $\mathbf{w}(X_s) = \mathbf{o}(X_s)$ for all $s \in Sl(X)$.

- $Gr(X) = Sl(X)$ iff there are no ungrounded slices, i.e., there exists $i \in [1, n]$ such that $l_i = 1$. Thus there exists i such that α_i is add. indec..

- According to Corollary 4.2.7, $\mathbf{w}(X_s) = \mathbf{o}(X_s)$ is true if and only if there exist $j_1 \neq j_2$ such that $\alpha_{j_1,s(j_1)}$ and $\alpha_{j_2,s(j_2)}$ are both infinite. In particular, for the top slice $s : j \mapsto l_j - 1$, there exist $j_1 \neq j_2$ such that $\alpha_{j_1,l_{j_1}-1}$ and $\alpha_{j_2,l_{j_2}-1}$ are both infinite, and therefore all exponents of α_{j_1} and α_{j_2} are infinite.

□

What is interesting with this result is that it can be extended to the cartesian product of any wqos:

Theorem 4.3.2. *Let $A_1, \ldots, A_n$ be a family of wqos. If there exist $i, j_1 \neq j_2 \in [1, n]$ such that $\mathbf{o}(A_i)$ is additively indecomposable, and $\mathbf{o}(A_{j_1})$ and $\mathbf{o}(A_{j_2})$ only have infinite exponents, then $\mathbf{w}(A_1 \times \cdots \times A_n) = \mathbf{o}(A_1) \otimes \cdots \otimes \mathbf{o}(A_n)$.*

Proof. According to Lemma 1.2.5,

$$\mathbf{w}(A_1 \times \cdots \times A_n) \leq \mathbf{o}(A_1) \otimes \cdots \otimes \mathbf{o}(A_n) \ .$$

On the other hand $A_1 \times \cdots \times A_n \leq_{\mathsf{aug}} \mathbf{o}(A_1) \times \cdots \times \mathbf{o}(A_n)$ through Lemma 2.1.8, thus:

$$\begin{aligned} \mathbf{w}(A_1 \times \cdots \times A_n) &\geq \mathbf{w}(\mathbf{o}(A_1) \times \cdots \times \mathbf{o}(A_n)) \\ &= \mathbf{o}(\mathbf{o}(A_1) \times \cdots \times \mathbf{o}(A_n)) \text{ according to Theorem 4.3.1,} \\ &= \mathbf{o}(A_1) \otimes \cdots \otimes \mathbf{o}(A_n) \ . \end{aligned}$$

□

Conclusion

Even though the width of the Cartesian product of wqos is non-functional, we managed to prove a lower bound (Theorem 4.1.3), and a sufficient condition for when width reaches mot (Theorem 4.3.2). As we will see in Chapter 7, this last theorem will be helpful to compute the ordinal invariants of an elementary family of wqos closed by Cartesian product.

5 - Multisets: One construction, two orderings, four ordinal invariants

In Section 5.2, we compute the width of the multiset embedding. We then deal with the multiset ordering in Section 5.3: we easily compute its height, however we prove that its width is not functional. We do not let that stop us, and introduce a fourth ordinal invariant, the friendly order type, in which the width of the multiset ordering is functional. We provide tools to compute this new ordinal invariant in Section 5.4.

5.1 . Orderings on the set of finite multisets

Formally multiset of elements of a set A can be described as a function $m : A \to \mathbb{N}$ whose support $\{ a \in A \mid f(A) > 0 \}$ is finite, which associates at each element its multiplicity. The multiset singleton $\langle\ a\ \rangle$ maps a to 1 and other elements to 0. The multiset union, intersection and subtraction are defined for any multiset $m, m' \in \mathsf{M}^{\diamond}(A)$ as:

$$\begin{aligned} m \cup m' &: a \mapsto m(a) + m'(a) \\ m \cap m' &: a \mapsto min(m(a), m'(a)) \\ m \setminus m' &: a \mapsto max(0, m(a) - m'(a)) \end{aligned}$$

For any $k \in \mathbb{N}$, $m \times k$ denotes the union of k copies of m. Let $\langle\ x_1, \ldots, x_n\ \rangle$ denote the union of the multiset singletons $\langle\ x_i\ \rangle$ for $i \in [1, k]$ (they do not have to be distinct). Let $|m|$ denote the number of elements of a multiset m, i.e., $|m| \stackrel{\mathsf{def}}{=} \Sigma_{a \in A} m(a)$.

Let us remind you of the two orderings classically defined on the set of finite multisets $M(A)$ of a quasi order A:

Definition 5.1.1 (Multiset embedding Weiermann (2009))**.** The *multiset embedding* $\mathsf{M}^{\diamond}(A)$, also known as the term ordering, is defined as:

$m \leq_{\diamond} m'$ iff there exists $f : m \to m'$ injective such that for any $x \in m$, $x \leq_A f(x)$.

Observe that $\mathsf{M}^{\diamond}(A)$ is isomorphic to the finite word embedding A^* divided by the equivalence relation "equal up to a permutation of letters".

Definition 5.1.2 (Multiset ordering Weiermann (1991))**.** The *multiset ordering* $\mathsf{M}^{\mathsf{r}}(A)$ of a wpo A is defined as:

$$m \leq_r m' \iff m = m' \text{ or } \forall x \in m \setminus (m \cap m'), \exists y \in m' \setminus (m \cap m'), x <_A y\ .$$

Recall that to adapt this definition to wqos, we have to divide A by its equivalence relation: $\mathsf{M}^{\mathsf{r}}(A) \stackrel{\mathsf{def}}{=} \mathsf{M}^{\mathsf{r}}(A_{/\equiv_A})$, this is why we will only deal with wpos in Section 5.3.

Remark 5.1.3. The main difference between these two orderings is that the multiset embedding gives priority to multiplicities of elements before their respective ordering (for any multisets m, m', $m \leq_\diamond m'$ implies $|m| \leq |m'|$), whereas in the multiset ordering one element can dominate many. For instance, $\langle\, 1\, \rangle \geq_r \langle\, 0, 0, 0\, \rangle$ in $\mathsf{M}^{\mathsf{r}}(\mathbb{N})$). This is why for any wpo A, $\mathsf{M}^\diamond(A) \leq_{\mathsf{aug}} \mathsf{M}^{\mathsf{r}}(A)$, as was observed by Aschenbrenner and Pong (2004).

Observe that if A is a linear ordering, then $\mathsf{M}^{\mathsf{r}}(A)$ is linear, while $\mathsf{M}^\diamond(A)$ is not as long as A has more than two elements: in $\mathsf{M}^\diamond(2)$, $\langle\, 1\, \rangle \perp_\diamond \langle\, 0, 0\, \rangle$.

Nonetheless, $\leq_\diamond$ and $\leq_r$ behave similarly on simple data structures.

Lemma 5.1.4 (Width of $M(\Gamma_k)$)**.** *Recall that Γ_k is the* wpo *that contains k incomparable elements. Then $\mathbf{w}(\mathsf{M}^\diamond(\Gamma_k)) = \mathbf{w}(\mathsf{M}^{\mathsf{r}}(\Gamma_k)) = \omega^{k-1}$.*

Proof. Since $\mathsf{M}^\diamond(\Gamma_1) = \mathsf{M}^{\mathsf{r}}(\Gamma_1) \cong \omega$, Equation (Alc-M-1) tells us that $\mathsf{M}^\diamond(\Gamma_k)$ and $\mathsf{M}^{\mathsf{r}}(\Gamma_k)$ are both isomorphic to the k-fold Cartesian product $\omega \times \cdots \times \omega$. And according to Proposition 4.1.9, $\mathbf{w}(\omega \times \cdots \times \omega) = \omega^{k-1}$. □

Ordinal invariants of the set of finite multisets Van der Meeren, Rathjen and Weiermann computed the mot of $\mathsf{M}^\diamond(A)$ (Van der Meeren et al. (2015); Weiermann (2009)) and $\mathsf{M}^{\mathsf{r}}(A)$ (Van der Meeren et al. (2015); Weiermann (1991)). Džamonja et al. (2020) computed the height of $\mathsf{M}^\diamond(A)$, with a partial result for the width of $\mathsf{M}^\diamond(A)$ (see Lemma 5.2.2).

Operation	M.O.T.	Height	Width
$\mathsf{M}^\diamond(A)$	$\omega^{\widehat{\mathbf{o}(A)}}$	$\mathbf{h}^*(A)$	$\mathbf{o}(\mathsf{M}^\diamond(A))$ when $\mathbf{o}(A) = \omega^\alpha$, $\alpha > 0$
$\mathsf{M}^{\mathsf{r}}(A)$	$\omega^{\mathbf{o}(A)}$	?	Non-functional

Table 5.1: Ordinal invariants of the finite multisets orderings (See Figure 1.3 for the definition of $\widehat{\alpha}$ and $\mathbf{h}^*$).

Observe that $\omega^{\mathbf{o}(A)} \leq \omega^{\widehat{\mathbf{o}(A)}}$, as one would expect since $\mathsf{M}^{\mathsf{r}}(A) \geq_{\mathsf{aug}} \mathsf{M}^\diamond(A)$. Furthermore, we expect that $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) \leq \mathbf{w}(\mathsf{M}^\diamond(A))$, while $\mathbf{h}(\mathsf{M}^{\mathsf{r}}(A)) \geq \mathbf{h}(\mathsf{M}^\diamond(A))$.

5.2 . Ordinal width of the multiset embedding

In this section we compute the width of $\mathsf{M}^\diamond(A)$ for any wqo A, which happens to be functional in the width of A:

Theorem 5.2.1 (Width of the multiset embedding). *For any* wqo A, $\mathbf{w}(\mathsf{M}^\diamond(A)) = \omega^{\widehat{\mathbf{o}(A)}-1}$.

This fits the partial result in Table 1.1, for when the width of the multiset embedding reaches its mot.

Lemma 5.2.2 (Džamonja et al. (2020)). *If* $\mathbf{o}(A)$ *is additively indecomposable,* $\mathbf{w}(\mathsf{M}^\diamond(A)) = \mathbf{o}(\mathsf{M}^\diamond(A)) = \omega^{\widehat{\mathbf{o}(A)}}$.

We focus for now on the set of finite multisets on a linear wqo, i.e., an ordinal. Let us treat first the case of successor ordinals.

Lemma 5.2.3. *For any successor ordinal* $\alpha = \beta+1$, $\mathbf{w}(\mathsf{M}^\diamond(\alpha)) \geq \mathbf{w}(\mathsf{M}^\diamond(\beta)) \cdot \omega$.

Proof. We denote with $\mathsf{M}^\diamond{}_{>k}(A)$ the subset $\{ m \in \mathsf{M}^\diamond(A) \mid |m| > k \}$ for any $k \in \mathbb{N}$ of $\mathsf{M}^\diamond(A)$ for any wqo A, for any $k < \omega$.

Let $m_n \stackrel{\text{def}}{=} \langle \beta \rangle \times n$ for any $n \in \mathbb{N}$. According to Equation (Res-w),

$$\begin{aligned}\mathbf{w}(\mathsf{M}^\diamond(\alpha)) &= \sup \{ \mathbf{w}(\mathsf{M}^\diamond(\alpha)_{\perp m}) + 1 \mid m \in \mathsf{M}^\diamond(\alpha) \} \\ &\geq \sup \{ \mathbf{w}(\mathsf{M}^\diamond(\alpha)_{\perp m_n}) + 1 \mid n \in \mathbb{N} \} \ .\end{aligned}$$

Let $M_k \stackrel{\text{def}}{=} \{ \langle \beta \rangle \times (n-k) \cup m \mid m \in \mathsf{M}^\diamond{}_{>k}(\beta) \}$ for $k \in [1,n]$. These subsets of $\mathsf{M}^\diamond(\alpha)$ are actually subsets of $\mathsf{M}^\diamond(\alpha)_{\perp m_n}$: for all $m \in M_k$, $m \perp m_n$ since $|m| > |m_n|$. Observe also that for any $k \in [1,n]$, $M_k \cong \mathsf{M}^\diamond(\beta)$.

Moreover, $(M_k)_{k\in[1,n]}$ is a family of quasi-incomparable* subsets of $\mathsf{M}^\diamond(\alpha)_{\perp m_n}$: for any $i < n$, for any finite $Y \subset M_1 \cup \cdots \cup M_i$, let $s(Y) = max\{|m|, m \in Y\}$. Observe that M_{i+1} contains $M_{i+1} \cap \mathsf{M}^\diamond{}_{>s(Y)}(\beta)$ which is incomparable to Y, and isomorphic to M_{i+1}.

Therefore, $\mathbf{w}(\mathsf{M}^\diamond(\alpha)_{\perp m_n}) \geq \mathbf{w}(M_n) + \cdots + \mathbf{w}(M_1) = \mathbf{w}(\mathsf{M}^\diamond(\beta)) \cdot n$ according to Lemma 2.3.2. Thus $\mathbf{w}(\mathsf{M}^\diamond(\alpha)) \geq \sup \{ \mathbf{w}(\mathsf{M}^\diamond(\beta) \cdot n + 1 \mid n \in \mathbb{N} \} = \mathbf{w}(\mathsf{M}^\diamond(\beta)) \cdot \omega$. ☐

Lemma 5.2.4. *For any infinite ordinal* α, $\mathbf{w}(\mathsf{M}^\diamond(\alpha)) = \mathbf{o}(\mathsf{M}^\diamond(\alpha))$.

Proof. We already know that $\mathbf{w}(\mathsf{M}^\diamond(\alpha)) \leq \mathbf{o}(\mathsf{M}^\diamond(\alpha))$. We prove the lower bound by induction on α:

- If α is additively indecomposable, see Lemma 5.2.2.

- If $\alpha = \beta + 1$, then according to Lemma 5.2.3,

$$\begin{aligned} \mathbf{w}(\mathsf{M}^\diamond(\alpha)) &\geq \mathbf{w}(\mathsf{M}^\diamond(\beta)) \cdot \omega \\ &= \mathbf{o}(\mathsf{M}^\diamond(\beta)) \cdot \omega && \text{by induction hypothesis,} \\ &= \omega^{\widehat{\beta}+1} = \omega^{\widehat{\beta+1}} = \mathbf{o}(\mathsf{M}^\diamond(\alpha)) && \text{according to Table 5.1.} \end{aligned}$$

- If $\alpha = \beta + \omega^\rho$ with $\beta, \omega^\rho < \alpha$ and $\rho > 0$, then according to the transformation equation Alc-M-3, $\mathsf{M}^\diamond(\alpha) \leq_{\mathsf{aug}} \mathsf{M}^\diamond(\beta) \cdot \mathsf{M}^\diamond(\omega^\rho)$. Hence according to Lemma 2.1.5 and Table 1.1,

$$\begin{aligned} \mathbf{w}(\mathsf{M}^\diamond(\alpha)) &\geq \mathbf{w}(\mathsf{M}^\diamond(\beta)) \odot \mathbf{w}(\mathsf{M}^\diamond(\omega^\rho)) \\ &= \mathbf{o}(\mathsf{M}^\diamond(\beta)) \odot \mathbf{o}(\mathsf{M}^\diamond(\omega^\rho)) && \text{by induction hypothesis,} \\ &= \omega^{\widehat{\beta}} \odot \omega^{\widehat{\omega^\rho}} = \omega^{\widehat{\alpha}} \\ &= \mathbf{o}(\mathsf{M}^\diamond(\alpha)) && \text{according to Table 5.1.} \ \square \end{aligned}$$

We can now prove that Lemma 5.2.4 generalizes to non-linear wqos.

Lemma 5.2.5. *If* $\mathbf{o}(A)$ *is infinite then* $\mathbf{w}(\mathsf{M}^\diamond(A)) = \mathbf{o}(\mathsf{M}^\diamond(A))$.

Proof. Let $\alpha = \mathbf{o}(A)$. Then $A \leq_{\mathsf{aug}} \alpha$ from Lemma 2.1.8, hence $\mathsf{M}^\diamond(A) \leq_{\mathsf{aug}} \mathsf{M}^\diamond(\alpha)$ according to Lemma 2.1.5 and Proposition 2.1.9. Thus

$$\mathbf{w}(\mathsf{M}^\diamond(\alpha)) \leq \mathbf{w}(\mathsf{M}^\diamond(A)) \leq \mathbf{o}(\mathsf{M}^\diamond(A)) \ .$$

Now $\mathbf{o}(\mathsf{M}^\diamond(A)) = \omega^{\widehat{\alpha}} = \mathbf{o}(\mathsf{M}^\diamond(\alpha))$ according to Table 5.1. Now with Lemma 5.2.4 $\mathbf{w}(\mathsf{M}^\diamond(\alpha)) = \mathbf{o}(\mathsf{M}^\diamond(\alpha))$, hence $\mathbf{w}(\mathsf{M}^\diamond(A)) = \mathbf{o}(\mathsf{M}^\diamond(A))$. □

We can also compute the width of $\mathsf{M}^\diamond(A)$ when A is a finite wqo:

Lemma 5.2.6. *If* $\mathbf{o}(A)$ *is finite, then* $\mathbf{w}(\mathsf{M}^\diamond(A)) = \omega^{\mathbf{o}(A)-1}$.

Proof. Let $k = \mathbf{o}(A)$. Then $\Gamma_k \leq_{\mathsf{aug}} A \leq_{\mathsf{aug}} k$, hence $\mathbf{w}(\mathsf{M}^\diamond(\Gamma_k)) \geq \mathbf{w}(\mathsf{M}^\diamond(A)) \geq \mathbf{w}(\mathsf{M}^\diamond(k))$ thanks to Lemma 2.1.5. According to Lemma 5.1.4, $\mathbf{w}(\mathsf{M}^\diamond(\Gamma_k)) = \omega^{k-1}$, and according to Lemma 5.2.3 applied $(k-1)$ times, $\mathbf{w}(\mathsf{M}^\diamond(k)) \geq \mathbf{w}(\mathsf{M}^\diamond(1)) \cdot \omega^{k-1} = \omega^{k-1}$. Therefore $\mathbf{w}(\mathsf{M}^\diamond(A)) = \omega^{k-1} = \omega^{\mathbf{o}(A)-1}$. □

This section's main result follows directly from Lemmas 5.2.5 and 5.2.6.

Proof of Theorem 5.2.1. If $\mathbf{o}(A)$ is finite, then $\widehat{\mathbf{o}(A)} - 1 = \mathbf{o}(A) - 1$. On the other hand, if $\mathbf{o}(A)$ is infinite, then $\widehat{\mathbf{o}(A)} - 1 = \widehat{\mathbf{o}(A)}$ (Equation (A.7)). □

Space	**o**	**h**	**w**
X_1	$\omega \cdot 2$	$\omega \cdot 2$	ω
X_2	$\omega \cdot 2$	$\omega \cdot 2$	ω
$\mathsf{M}^{\mathsf{r}}(X_1)$	$\omega^{\omega \cdot 2}$	$\omega^{\omega \cdot 2}$	ω^{ω}
$\mathsf{M}^{\mathsf{r}}(X_2)$	$\omega^{\omega \cdot 2}$	$\omega^{\omega \cdot 2}$	$\omega^{\omega \cdot 2}$

5.3 . Ordinal height and width of the multiset ordering

The height of $\mathsf{M}^{\mathsf{r}}(A)$ follow the same pattern as its mot.

Theorem 5.3.1 (Height of the multiset ordering)**.** *Let A be a* wpo*. Then* $\mathbf{h}(\mathsf{M}^{\mathsf{r}}(A)) = \omega^{\mathbf{h}(A)}$.

Proof. Observe that the multiset ordering of any linear ordering is also linear. Thus, for any ordinal α, $\mathsf{M}^{\mathsf{r}}(\alpha)$ is isomorphic to ω^{α} (the function $\langle\, x_1, \ldots, x_n \,\rangle \mapsto \omega^{x_1} \oplus \cdots \oplus \omega^{x_n}$ is an isomorphism).

According to Lemma 2.1.8, $A \geq_{\mathsf{st}} \mathbf{h}(A)$, and thus $\mathsf{M}^{\mathsf{r}}(A) \geq_{\mathsf{st}} \mathsf{M}^{\mathsf{r}}(\mathbf{h}(A)) \cong \omega^{\mathbf{h}(A)}$ (Proposition 2.1.9). Therefore $\mathbf{h}(\mathsf{M}^{\mathsf{r}}(A)) \geq \omega^{\mathbf{h}(A)}$ according to Lemma 2.1.5. See the proof of the upper bound in Appendix 5.5. □

The width of the multiset ordering is harder to compute: as we have seen in Example 1.3.6 (which we can now prove), $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A))$ is not functional in the ordinal invariants of A.

Example 1.3.6 (Non-functionality example: Multiset ordering)**.** *As in Example 1.3.5, let* $X_1 \stackrel{\mathsf{def}}{=} H + \omega$ *and* $X_2 \stackrel{\mathsf{def}}{=} H + H$*. Then* $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(X_1)) \neq \mathbf{w}(\mathsf{M}^{\mathsf{r}}(X_2))$.

Proof. Since $\mathsf{M}^{\mathsf{r}}(H) \geq_{\mathsf{st}} \mathsf{M}^{\mathsf{r}}(\Gamma_n)$, then $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(H)) \geq \omega^{n-1}$ for all $n < \omega$ according to Lemmas 5.1.4 and 2.1.5. On the other hand, $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(H)) \leq \mathbf{o}(\mathsf{M}^{\mathsf{r}}(H)) = \omega^{\omega}$ according to Lemma 1.2.5 and Table 1.1. Hence $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(H)) = \omega^{\omega}$. Therefore according to Equation (Alc-M-2) and Table 1.1, $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(X_1)) = \mathbf{w}(\mathsf{M}^{\mathsf{r}}(H)) \odot \mathbf{w}(\mathsf{M}^{\mathsf{r}}(\omega)) = \omega^{\omega} \odot 1 = \omega^{\omega}$ and $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(X_2)) = \mathbf{w}(\mathsf{M}^{\mathsf{r}}(H)) \odot \mathbf{w}(\mathsf{M}^{\mathsf{r}}(H)) = \omega^{\omega} \odot \omega^{\omega} = \omega^{\omega \cdot 2}$. □

Fortunately, we uncovered a new ordinal invariant, defined similarly to the usual invariants, in which the width of the multiset ordering is functional.

Definition 5.3.2 (Friendly order type)**.** A bad sequence is *open-ended* if it is empty or of the form sx where s is an open-ended sequence and x has a "friend" [a] in the residual $A_{\not\geq s}$, i.e., an element incomparable to x. For any wpo A, let $\mathsf{Bad}_\perp(A)$ be the subtree of $\mathsf{Bad}(A)$ which contains all open-ended bad sequences. As $\mathsf{Bad}_\perp(A)$ is a substructure of $\mathsf{Bad}(A)$, it has a rank that we denote by $\mathbf{o}_\perp(A)$ the *friendly order type* of A (or fot).

This definition can be expressed as the following residual equation:

$$\mathbf{o}_\perp(A) = \sup_{x \in A, A_{\perp x} \neq \emptyset} (\mathbf{o}_\perp(A_{\not\geq x}) + 1) \tag{Res-f}$$

Theorem 5.3.3. *For any* wpo A*,* $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) = \omega^{\mathbf{o}_\perp(A)}$

Proof. See Section 5.5. The proof of Theorem 5.3.3 is quite technical, and relies on the notion of quasi-incomparable subsets. □

5.4 . Computing the friendly order type

Friendly order type behaves similarly to mot. It is not unusual to have fot coincides with mot, for instance $\mathbf{o}_\perp(\omega \sqcup \omega) = \mathbf{o}(\omega \sqcup \omega)$ (we will prove that later with Theorem 5.4.8).

To bring this new ordinal invariant closer to familiar grounds, we bound the fot of a wpo A with the mot of a special subset of A, the stripped subset.

Definition 5.4.1 (Stripped subset)**.** The *stripped* subset of a wpo A, denoted by $str(A)$, is A without its friendless elements:

$$str(A) \stackrel{\mathsf{def}}{=} \{ x \in A \mid A_{\perp x} \neq \emptyset \} .$$

Since $\mathsf{Bad}_\perp(A)$ is a subtree of $\mathsf{Bad}(\mathrm{str}(A))$, we know that $\mathbf{o}_\perp(A) \leq \mathbf{o}(\mathrm{str}(A))$. Here is an example where this inequality is strict:

Example 5.4.2. *Let* $A = \omega \sqcup \{\clubsuit\}$*. Here* $str(A) = A$*, so* $\mathbf{o}(str(A)) = \omega + 1$*. However, in* $\mathsf{Bad}_\perp(A)$*, the singleton* $\clubsuit$ *has rank* 0*, and the singleton* n *for any* $n \in \omega$ *has rank* n*. Therefore* $\mathbf{o}_\perp(A) = \omega < \mathbf{o}(str(A))$*.*

Let us show that $\mathbf{o}(\mathrm{str}(A))$ also appears in a lower bound on $\mathbf{o}_\perp(A)$, by introducing an alternative characterisation of fot as the mot of a specific subset.

Definition 5.4.3 (Friendly subset)**.** A subset A' of A is friendly if there exist a linearisation function $\ell : A' \to \mathbf{o}(A')$ such that for any bad sequence $s = x_1, \ldots x_n$ in A' verifying $\ell(x_1) > \cdots > \ell(x_n)$, s is open-ended. We say that ℓ witnesses the *friendly condition.*

[a] Can one be friend with one's superior or inferior? No. Your true friends are those you cannot (and do not have to) compare yourselves with.

Observe that every friendly subset of A is a substructure of $\mathrm{str}(A)$.
For any ordinal α, let

$$\delta(\alpha) \stackrel{\mathsf{def}}{=} \begin{cases} \alpha \text{ if } \alpha \text{ is limit,} \\ \gamma + \lfloor n/2 \rfloor \text{ if } \alpha = \gamma + n \text{ with } \gamma \text{ limit and } n < \omega. \end{cases}$$

Theorem 5.4.4 (Alternative characterisation of $\mathbf{o}_\perp(A)$)**.** *Let A be a* wpo*. There exists a friendly subset A' of A which maximizes $\mathbf{o}(A')$, and $\mathbf{o}_\perp(A) = \mathbf{o}(A')$. Furthermore, $\delta(\mathbf{o}(str(A))) \leq \mathbf{o}_\perp(A) \leq \mathbf{o}(str(A))$.*

Proof. See proof in Section 5.5. □

Example 5.4.5 (Following on Example 1.3.6)**.** *Remember that $H \stackrel{\mathsf{def}}{=} \Sigma_{n<\omega}\Gamma_n$. Thus $str(H) = \Sigma_{2\leq n<\omega}\Gamma_n$, and $\mathbf{o}(str(H)) = \mathbf{o}(H) = \omega$. Consider $X_1 = H + H$ and $X_2 = H + \omega$. Observe that $str(X_1) = str(H) + str(H)$ whereas $str(X_2) = str(H)$. Therefore, according to Theorem 5.4.4, $\mathbf{o}_\perp(X_1) = \omega \cdot 2$ and $\mathbf{o}_\perp(X_2) = \omega$.*

Corollary 5.4.6. *For any* wpo *A, if $\mathbf{o}(A)$ is limit and $\mathbf{o}(str(A)) = \mathbf{o}(A)$, then $\mathbf{o}_\perp(A) = \mathbf{o}(A)$.*

The conditions in Corollary 5.4.6 are easily satisfied:

Lemma 5.4.7. *For any* wpo *A, if $\mathbf{w}(A) = \mathbf{o}(A) > 1$, then $\mathbf{o}(str(A)) = \mathbf{o}(A)$*

Proof. Observe that if $\mathbf{w}(A) > 1$, then $\mathbf{w}(str(A)) = \mathbf{w}(A)$. And $\mathbf{w}(str(A)) \leq \mathbf{o}(str(A)) \leq \mathbf{o}(A)$ according to Lemmas 1.2.5 and 2.1.5, we conclude from there. □

Like the usual ordinal invariants, the fot can be computed compositionally for some basic operations on wpos. Rather than give exact computations, we choose to give sufficient conditions for when friendly and maximal order types coincides.

Theorem 5.4.8. *For any* wqo *A, B*

1. $\mathbf{o}_\perp(A + B) = \mathbf{o}_\perp(A) + \mathbf{o}_\perp(B)$,
2. $\mathbf{o}_\perp(A \sqcup B) = \mathbf{o}(A \sqcup B)$ *if $\mathbf{o}(A)$ and $\mathbf{o}(B)$ are limit ordinals.*
3. $\mathbf{o}_\perp(A \times B) = \mathbf{o}(A \times B)$ *if $\mathbf{o}(A)$ and $\mathbf{o}(B)$ are limit ordinals.*
4. $\mathbf{o}_\perp(A^*) = \mathbf{o}(A^*)$ *if $\mathbf{o}(A) \leq 2$.*
5. $\mathbf{o}_\perp(\mathsf{M}^\diamond(A)) = \mathbf{o}(\mathsf{M}^\diamond(A))$
6. $\mathbf{o}_\perp(\mathsf{M}^\mathsf{r}(A)) = \mathbf{o}(\mathsf{M}^\mathsf{r}(A))$ *if $str(A) \neq \emptyset$. If A is a linear ordering then $\mathbf{o}_\perp(\mathsf{M}^\mathsf{r}(A)) = 0$.*

7. $\mathbf{o}_\perp(\mathcal{T}(A)) = \mathbf{o}(\mathcal{T}(A))$

8. $\mathbf{o}_\perp(\mathsf{P_f}(A)) = \mathbf{o}(\mathsf{P_f}(A))$ *if* $\mathbf{w}(A) = \mathbf{o}(A)$ *is limit.*

Proof. 1. For any sequences s_A, s_B in $\mathsf{Bad}_\perp(A), \mathsf{Bad}_\perp(B)$, the concatenation $s_B s_A$ is a sequence of $\mathsf{Bad}_\perp(A+B)$. Furthermore, any sequence of $\mathsf{Bad}_\perp(A+B)$ is of this form.

2. Observe that $str(A \sqcup B) = A \sqcup B$ since A and B are not empty. And $\mathbf{o}(A \sqcup B)$ is limit when $\mathbf{o}(A)$ and $\mathbf{o}(B)$ are (Table 1.1), hence $A \sqcup B$ satisfies the conditions of Corollary 5.4.6.

3. Observe that $str(A \times B) = (A \times B) \setminus (a, b)$ if A and B each have one unique minimal element a and b, otherwise $str(A \times B) = (A \times B)$. If $\mathbf{o}(A)$ and $\mathbf{o}(B)$ are infinite limit ordinals then in both cases $\mathbf{o}(str(A \times B)) = \mathbf{o}(A \times B)$ which is a limit ordinal (Table 1.1), hence $A \times B$ satisfies the conditions of Corollary 5.4.6.

4. If $\mathbf{o}(A) \leq 2$, then A^* satisfies the conditions of Corollary 5.4.6 and Lemma 5.4.7 according to Table 1.1.

5. Observe that $str(\mathsf{M}^\diamond(A)) = \mathsf{M}^\diamond(A) \setminus \{\emptyset, \langle\ a\ \rangle\}$ or $\mathsf{M}^\diamond(A) \setminus \{\emptyset\}$ depending on whether A has a unique minimal element a. Thus $\mathbf{o}(str(\mathsf{M}^\diamond(A))) = \mathbf{o}(\mathsf{M}^\diamond(A))$ since $\mathbf{o}(\mathsf{M}^\diamond(A))$ is limit (Equation (A.7) and Table 5.1). We conclude with Corollary 5.4.6.

6. If A is linear then $\mathsf{M}^\mathsf{r}(A)$ is also linear (Remark 5.1.3), therefore $\mathbf{o}_\perp(\mathsf{M}^\mathsf{r}(A))$ is null. Otherwise, let $x, y \in A$ such that $x \perp y$. Then for all $m \in \mathsf{M}^\mathsf{r}(A)$, $m \cup \langle\ x\ \rangle \perp m \cup \langle\ y\ \rangle$, hence $str(\mathsf{M}^\mathsf{r}(A)) \geq_{\mathsf{st}} \{\ m \cup \langle\ x\ \rangle \mid m \in \mathsf{M}^\mathsf{r}(A)\ \} \cong \mathsf{M}^\mathsf{r}(A)$. Hence $\mathbf{o}(str(\mathsf{M}^\diamond(A))) = \mathbf{o}(\mathsf{M}^\mathsf{r}(A)) = \omega^{\mathbf{o}(A)}$ a limit ordinal according to Table 5.1. Therefore $\mathbf{o}_\perp(\mathsf{M}^\mathsf{r}(A)) = \mathbf{o}(\mathsf{M}^\mathsf{r}(A))$ according to Corollary 5.4.6.

7. $\mathcal{T}(A)$ satisfies the conditions of Corollary 5.4.6 and Lemma 5.4.7 according to Table 1.1.

8. As we will see in next chapter, $\mathbf{w}(\mathsf{P_f}(A)) = \mathbf{o}(\mathsf{P_f}(A)) = 2^{\mathbf{o}(A)}$ when $\mathbf{w}(A) = \mathbf{o}(A)$. If $\mathbf{o}(A)$ is limit then so is $2^{\mathbf{o}(A)}$ (Proposition A.0.17), hence $\mathsf{P_f}(A)$ satisfies the conditions of Corollary 5.4.6 and Lemma 5.4.7. $\square$

5.5 . Appendix

Proof of Theorem 5.3.1

We write $m \overset{\not\emptyset}{\leq_r} m'$ (resp. $m \overset{\not\emptyset}{<} m'$, $m \overset{\not\emptyset}{\perp} m'$) when $m \cap m' \neq \emptyset$ and $m \leq_r m'$ (resp $m < m'$, $m \perp m'$). With these new notations, the multiset ordering can be reformulated as follows

Definition 5.5.1 (Multiset ordering (reformulated))**.** $\mathsf{M}^\mathsf{r}(A) = (M(A), \leq_r)$ is ordered with the *multiset ordering*: $m \leq_r m'$ iff there exists m_1, m'_1, m_2 such that $m = m_1 \cup m_2$, $m' = m'_1 \cup m_2$, and $m_1 \overset{\not\emptyset}{<} m'_1$.

Proof of Theorem 5.3.1. We prove the upper bound by induction on $\mathbf{h}(A)$.

If $\mathbf{h}(A) = 0$ then $A = \emptyset$ and $\mathbf{h}(\mathsf{M}^\mathsf{r}(\emptyset)) = 1 = \omega^0$.

Suppose that A is not empty. For any non-empty multiset $m \in \mathsf{M}^\mathsf{r}(A)$, the residual $\mathsf{M}^\mathsf{r}(A)_{<m}$ can be partitioned as follows:

$$\mathsf{M}^\mathsf{r}(A)_{<m} = \bigcup_{m_1+m_2=m, m_1\neq\emptyset} \left\{ m' + m_2 \;\middle|\; m' \overset{\not\emptyset}{<} m_1 \right\} .$$

Note that this union is a partition of the support of $\mathsf{M}^\mathsf{r}(A)_{<m}$, it does not say anything on the order between the elements of the subsets in the union.

For any non-empty multiset m, we define $S_m \overset{\text{def}}{=} (\cap_{x\in m} A_{\not\geq x}) \cap (\cup_{x\in m} A_{<x})$ a subset of A. Thus for any multiset m' in $\mathsf{M}^\mathsf{r}(A)$, $m' \overset{\not\emptyset}{<} m$ iff $m' \in \mathsf{M}^\mathsf{r}(S_m)$. Therefore:

$$\mathsf{M}^\mathsf{r}(A)_{<m} = \bigcup_{m_1+m_2=m, m_1\neq\emptyset} \{ m' + m_2 \mid m' \in \mathsf{M}^\mathsf{r}(S_{m_1}) \} .$$

Observe that $\mathbf{h}(S_{m_1}) < \mathbf{h}(A)$ by definition of S_{m_1}. Hence by induction hypothesis $\mathbf{h}(\mathsf{M}^\mathsf{r}(S_{m_1})) \leq \omega^{\mathbf{h}(S_{m_1})} < \omega^{\mathbf{h}(A)}$. Moreover, $\omega^{\mathbf{h}(A)}$ is additively indecomposable. Hence according to Lemma 2.2.5:

$$\mathbf{h}(\mathsf{M}^\mathsf{r}(A)_{<m}) \leq \bigoplus_{m_1+m_2=m, m_1\neq\emptyset} \mathbf{h}(\mathsf{M}^\mathsf{r}(\cup_{x\in m_1} A_{<x})) < \omega^{\mathbf{h}(A)} .$$

Therefore $\mathbf{h}(\mathsf{M}^\mathsf{r}(A)) \leq \omega^{\mathbf{h}(A)}$ according to Equation (Res-h). □

Proof of Theorem 5.3.3

First we prove intermediary lower and upper bounds on the width of the multiset ordering.

Lemma 5.5.2. *Let A be a* wpo*. Then*

$$\mathbf{w}(\mathsf{M}^\mathsf{r}(A)) \geq \sup_{x\in A, n<\omega} \mathbf{w}(\mathsf{M}^\mathsf{r}(A)_{\perp\langle x \rangle}) \cdot n + 1$$

Proof. This proof follows the same structure as the proof of Lemma 5.2.3: We study the residual of $\mathsf{M}^{\mathsf{r}}(A)$ which contains every element incomparable to some multiset of the form $\langle\, x\,\rangle \times n$, and slice this residual into a family of quasi-incomparable* subsets.

According to Equation (Res-w),

$$\begin{aligned} w(\mathsf{M}^{\mathsf{r}}(A)) &= \sup_{m\in\mathsf{M}^{\mathsf{r}}(A)} \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp m}) + 1 \\ &\geq \sup_{x\in A, n<\omega} \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\, x\,\rangle\times n}) + 1 \ . \end{aligned}$$

For all $k \in [1, n]$, let $M_k = \left\{ \langle\, x\,\rangle \times (n-k) \cup m \;\middle|\; m \in \mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\, x\,\rangle} \right\}$. Observe that $M_k \cong \mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\, x\,\rangle}$ for any $k \in [1, n]$, and for all $m \in M_k$, $m \perp \langle\, x\,\rangle \times n$. We claim that $(M_k)_{k\in[1,n]}$ is a family of quasi-incomparable* subsets of $\mathsf{M}^{\mathsf{r}}(A)_{\perp(\langle\, x\,\rangle\times n)}$: Let $i < n$ and Y a finite subset of $M_1 \cup \cdots \cup M_i$. We define m_Y and M'_{i+1} as

$$\begin{aligned} m_Y &\stackrel{\mathsf{def}}{=} \bigcup_{j\leq i} \bigcup_{m\in(M_j\cap Y)} (m \setminus (\langle\, x\,\rangle \times (n-j))) \ , \\ M'_{i+1} &\stackrel{\mathsf{def}}{=} \left\{ \langle\, x\,\rangle \times (n-i-1) \cup m_Y \cup m \;\middle|\; m \in \mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\, x\,\rangle} \right\} \ . \end{aligned}$$

Observe that M'_{i+1} is an isomorphic subset of M_{i+1}, and $Y \perp M'_{i+1}$.

Therefore according to Lemma 2.3.2, $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp(\langle\, x\,\rangle\times n)}) \geq \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\, x\,\rangle}) \cdot n$. □

Lemma 5.5.3. *Let A be a* wpo*. Then*

$$\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) \leq \sup_{x\in A, n<\omega} \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\, x\,\rangle}) \otimes n + 1$$

Proof. By definition, for any multisets $m, m' \in \mathsf{M}^{\mathsf{r}}(A)$, $m \perp m'$ means that $m \neq m'$ and there exists m_1, m'_1, m_2 such that $m = m_1 \cup m_2$, $m' = m'_1 \cup m_2$ and $m_1 \overset{\not\supset}{\perp} m'_1$.

Therefore, the residual $\mathsf{M}^{\mathsf{r}}(A)_{\perp m}$ can be partitioned as an augmentation of a disjoint union:

$$\mathsf{M}^{\mathsf{r}}(A)_{\perp m} \geq_{\mathsf{aug}} \bigsqcup_{m_1+m_2=m, m_1\neq\emptyset} \left\{ m'_1 + m_2 \;\middle|\; m' \in \mathsf{M}^{\mathsf{r}}(A), m'_1 \overset{\not\supset}{\perp} m_1 \right\} \ ,$$

which can be reformulated into

$$\mathsf{M}^{\mathsf{r}}(A)_{\perp m} \geq_{\mathsf{aug}} \bigsqcup_{m_1\subseteq m, m_1\neq\emptyset} \mathsf{M}^{\mathsf{r}}(A)_{\overset{\not\supset}{\perp} m_1}$$

where $\mathsf{M}^{\mathsf{r}}(A)_{\overset{\not\supset}{\perp} m_1}$ is the residual $\left\{ m' \in \mathsf{M}^{\mathsf{r}}(A) \;\middle|\; m' \overset{\not\supset}{\perp} m_1 \right\}$.

Let us observe this residual: $m' \overset{\not\emptyset}{\perp} m_1$ means that m' and m_1 are disjoint and there exists $x \in m_1$ such that for all $y' \in m'$, $x \not\leq y'$, and there exists $x' \in m'$ such that for all $y \in m_1$, $x' \not\leq y$. In particular $x' \not\leq x$. Hence $m' \overset{\not\emptyset}{\perp} m_1$ implies there exists $x \in m_1$ such that $\langle\ x\ \rangle \overset{\not\emptyset}{\perp} m'$, which is equivalent to $\langle\ x\ \rangle \perp m'$. Therefore the support of $\mathsf{M}^{\mathsf{r}}(A)_{\overset{\not\emptyset}{\perp} m_1}$ is included in a union on $x \in m_1$ of residuals $\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle}$. With an augmentation we get a disjoint union:

$$\mathsf{M}^{\mathsf{r}}(A)_{\overset{\not\emptyset}{\perp} m_1} \leq_{\mathsf{st}} \geq_{\mathsf{aug}} \bigsqcup_{x \in m_1} \mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle} \ .$$

Hence according to Table 1.1, $\mathsf{M}^{\mathsf{r}}(A)_{\perp m} \leq \bigoplus_{m_1 \subseteq m, m_1 \neq \emptyset} \bigoplus_{x \in m_1} \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle})$.

Let $x \in m$ such that $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle})$ is maximal. Then $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp m}) \leq \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle}) \otimes n$ for some $n < \omega$. Hence according to Equation (Res-w),

$$\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) = \sup_{m \in \mathsf{M}^{\mathsf{r}}(A)} \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp m}) + 1 \leq \sup_{x \in A, n < \omega} \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle}) \otimes n + 1 \ .$$

□

The bounds provided in Lemmas 5.5.2 and 5.5.3 actually match. Furthermore, they can be reformulated in such a way that the residual on $\mathsf{M}^{\mathsf{r}}(A)$ boils down to a residual on A:

Lemma 5.5.4. *For any non-linear* wpo A*,*

$$\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) = \sup \{\ \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A_{\not\geq x})) \cdot \omega \mid x \in A, A_{\perp x} \neq \emptyset\ \} \ . \qquad \text{(W)}$$

Proof. For any ordinal α, $\sup_{n<\omega}(\alpha \cdot n + 1) = \sup_{n<\omega}(\alpha \otimes n + 1) = \alpha \cdot \omega$. Hence according to Lemmas 5.5.2 and 5.5.3, $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) = \sup_{x \in A}(\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle}) \cdot \omega)$.

Let $x \in A$. If $A_{\perp x} = \emptyset$, then $\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle} = \emptyset$. Otherwise let $y \in A_{\perp x}$. Observe that, for any $m \in \mathsf{M}^{\mathsf{r}}(A_{\not\geq x})$, $m \cup \langle\ y\ \rangle \perp \langle\ x\ \rangle$. Hence

$$\{\ \langle\ y\ \rangle \cup m \mid m \in \mathsf{M}^{\mathsf{r}}(A_{\not\geq x})\ \} \leq_{\mathsf{st}} \mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle} \leq_{\mathsf{st}} \mathsf{M}^{\mathsf{r}}(A_{\not\geq x}) \ .$$

Therefore $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle}) = \mathbf{w}(\mathsf{M}^{\mathsf{r}}(A_{\not\geq x}))$ if $A_{\perp x} \neq \emptyset$. Otherwise $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)_{\perp\langle\ x\ \rangle})$ is null. □

Proof of Theorem 5.3.3. If A is linear, $\mathsf{Bad}_{\perp}(A)$ only contains the empty sequence, hence $\mathbf{o}_{\perp}(A) = 0$ and $\mathbf{w}(Mulr(A) = 1$. Otherwise, observe that Equation (W) is quite similar to Equation (Res-f) in its structure. Thus $\mathbf{w}(\mathsf{M}^{\mathsf{r}}(A)) = \omega^{\mathbf{o}_{\perp}(A)}$ follows directly from Equation (W). □

Proof of Theorem 5.4.4

Lemma 5.5.5. *For any* wpo A*, for any linearisation function* $\ell : str(A) \to \mathbf{o}(str(A))$*, there exists a friendly subset* A' *such that* ℓ *restricted to* A' *verifies the friendly condition, and* $\mathbf{o}(A') \geq \delta(\mathbf{o}(str(A)))$*.*

Proof. We claim that for any $\beta \leq \mathbf{o}(str(A))$, there exists $A_\beta \subseteq \ell^{-1}(\{ \gamma : \gamma < \beta \})$ a friendly subset of A where ℓ restricted to A_β verifies the friendly condition, such that $\mathbf{o}(A_\beta) \geq \delta(\beta)$. In this proof, when we say that a subset is friendly, it is always implied that ℓ restricted to this subset witnesses the friendly condition.

We build the subsets $(A_\beta)_{\beta \leq \mathbf{o}(str(A))}$ as follows:

- $A_0 = \emptyset$,
- For γ limit, $A_\gamma = \bigcup_{\beta<\gamma} A_\beta$,
- For any β, $A_{\beta+1} = A_\beta \cup \ell^{-1}(\beta)$ if friendly, otherwise $A_{\beta+1} = A_\beta$.

First observe that A_β is friendly for any $\beta \leq \mathbf{o}(str(A))$. Indeed, A_0 is friendly, and since for any $\beta < \beta'$, $A_\beta \subseteq A_{\beta'}$, then the union $\bigcup_{\beta<\gamma} A_\beta$ for γ limit is friendly by induction.

Let us prove the claim $\mathbf{o}(A_\beta) \geq \delta(\beta)$, by showing that for any $\beta + 2 \leq \mathbf{o}(str(A))$, we have $\mathbf{o}(A_{\beta+2}) > \mathbf{o}(A_\beta)$. Let $x = \ell^{-1}(\beta')$ and $x' = \ell^{-1}(\beta' + 1)$. Assume for the sake of contradiction that $A_{\beta+2} = A_\beta$. This means that neither $A_\beta \cup \{x\}$ nor $A_\beta \cup \{x'\}$ are friendly. Hence there exists $y, y' \in A_\beta$ such that for any $z \in A$, we have $z \perp y \implies z \geq x$ and $z \perp y' \implies z \geq x$. Now because of ℓ we know that $x \not\geq x'$ and $y, y' \not\geq x, x'$. Since $y, y' \in str(A)$, then $A_{\perp y}$ and $A_{\perp y'}$ are both non-empty, so actually $x \perp y$ and $x' \perp y'$. And since $x \not\geq x'$, we know $y' < x$. Therefore $x \perp x'$, hence $y < x'$. Which leads to a contradiction on the relationship between y and y'. □

For any friendly subset A', $\mathbf{o}(A') \leq \mathbf{o}(\mathrm{str}(A))$, and there exist A' such that $\mathbf{o}(A') \geq \delta(\mathbf{o}(\mathrm{str}(A)))$. Therefore there exists a friendly subset A' which maximizes $\mathbf{o}(A')$.

Proof of Theorem 5.4.4. We say that a bad sequence $x_1, \ldots x_n$ respects a linearisation function ℓ when $\ell(x_1) > \cdots > \ell(x_n)$. Let A' be a friendly subset of A and $\ell : A' \to \mathbf{o}(A')$ a linearisation function that verifies the friendly condition. Observe that $\mathsf{Bad}(A')$ restricted to sequences that respect ℓ has for rank $\mathbf{o}(A')$, and is embedded in $\mathsf{Bad}_\perp(A)$. Hence $\mathbf{o}_\perp(A) \geq \mathbf{o}(A')$.

We prove the upper bound by induction on $\mathbf{o}_\perp(A)$. If $\mathbf{o}_\perp(A) = 0$ then the only friendly subset of A is the empty set. Now suppose that $\mathbf{o}_\perp(A) > 0$. For any $x \in str(A)$, by induction hypothesis on $A_{\not\geq x}$, there exists a friendly subset A' of $A_{\not\geq x}$, with a linearisation function $\ell : A' \to \mathbf{o}(A')$ which verifies the friendly condition, such that $\mathbf{o}(A') \geq \mathbf{o}_\perp(A_{\not\geq x})$. We extend ℓ to the subset

$A' \cup \{x\}$ of A, such that $\ell(x) = \mathbf{o}(A')$. Now ℓ is a linearisation function of $A' \cup \{x\}$ which verifies the friendly condition, therefore $\mathbf{o}(A' \cup \{x\})$ is a friendly subset of A and $\mathbf{o}(A' \cup \{x\}) > \mathbf{o}_\perp(A_{\not\succeq x})$. Let A' be a friendly subset of A which maximizes $\mathbf{o}(A')$. Then for any $x \in \mathit{str}(A)$, $\mathbf{o}_\perp(A_{\not\succeq x}) < \mathbf{o}(A')$. Therefore $\mathbf{o}_\perp(A) \leq \mathbf{o}(A')$ according to Equation (Res-f). $\square$

6 - Measuring the finite powerset

This chapter and the next chapter both comes from Abriola et al. (2023). This chapter is a result of joint work, where my contribution amount to bug-fixing. However, Chapter 7 is mostly my work.

In Section 6.1 we prove that all three ordinal invariants of the finite powerset are not functional. Then in Sections 6.2 and 6.3, we provide lower and upper bounds for mot and height. In Section 6.4, we give a lower bound on the width of $\mathsf{P_f}(A)$, and prove that there is no upper bound depending only on $\mathbf{w}(A)$. For a proof of the tightness of these bounds, refer to Abriola et al. (2023) from which this chapter is extracted.

6.1 . All ordinal invariants of the finite powerset are non functional

We recall the definition of Hoare's ordering on $\mathsf{P_f}(A)$ for A a wqo.

$$S \sqsubseteq_{\mathcal{H}} S' \iff \forall x \in S,\ \exists y \in S',\ x \leq_A y\ .$$

Remember that even when A is a wpo, $\mathsf{P_f}(A)$ may not be (see Example 1.1.1). Fortunately, $\mathsf{P_f}(A)$ is quasi-isomorphic to $\mathsf{Ant}(A)$, the set of finite antichains ordered with Hoare embedding, through the relation $S \mapsto \max_{\leq_A} S$ for $S \in \mathsf{P_f}(A)$. If A is a wpo then $\mathsf{Ant}(A)$ is a wpo too.

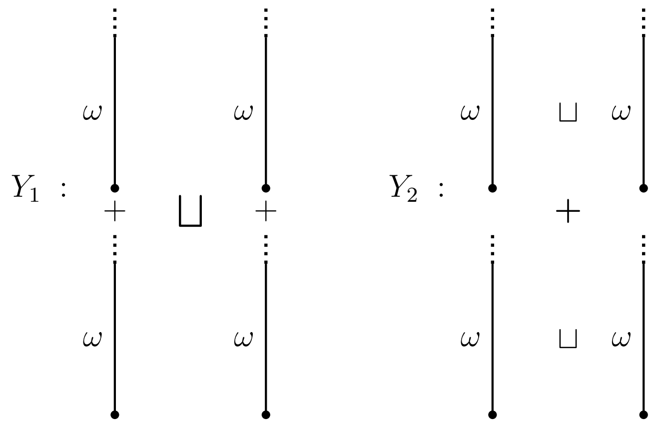
Similarly, observe that the finite powerset of a linear wqo is wpo-isomorphic to a linear wqo: $\mathsf{P_f}(\alpha) \cong 1 + \alpha$ for any ordinal α.

We are now able to measure the wqos in Example 1.3.8, which proves that the ordinal invariants of the finite powerset are non functional – all three of them !

Example 1.3.8 (Non-functionality example: Finite powerset)**.** *Consider* $Y_1 = (\omega + \omega) \sqcup (\omega + \omega)$ *and* $Y_2 = (\omega \sqcup \omega) + (\omega \sqcup \omega)$. *These two* wqos *have the same ordinal invariants, but* $\mathsf{P_f}(Y_1)$ *and* $\mathsf{P_f}(Y_2)$ *differ on all three ordinal invariants.*

Proof of Example 1.3.8. Thanks to Equation (Alc-Pf-2) and Equation (Alc-Pf-1) we deduce that $\mathsf{P_f}(Y_1) \cong \omega \cdot 2 \times \omega \cdot 2$, and that $\mathsf{P_f}(Y_2) \cong (\omega \times \omega) + (\omega \times \omega)$. As a consequence, with Theorem 4.2.1 we obtain the following table of ordinal invariants.

□



Space	o	h	w
Y_1	$\omega \cdot 4$	$\omega \cdot 2$	2
Y_2	$\omega \cdot 4$	$\omega \cdot 2$	2
$\mathsf{P_f}(Y_1)$	$\omega^2 \cdot 4$	$\omega \cdot 3$	$\omega \cdot 3$
$\mathsf{P_f}(Y_2)$	$\omega^2 \cdot 2$	$\omega \cdot 2$	ω

As for Cartesian product and finite multiset construction, non-functionality will not stop us. We can leverage the structural lemmas from Section 2.5 to bounds the ordinal invariants of the finite powerset.

6.2 . Bounds on the maximal order type of $\mathsf{P_f}(A)$

Let A be a wqo. It is clear that $1 + A \leq_{\mathsf{st}} \mathsf{P_f}(A)$ modulo isomorphism through the map $1 \mapsto \emptyset$ and $x \mapsto \{ x \}$. As a consequence of Table 1.1 and Lemma 2.1.5, we immediately conclude that the following lower bound holds

$$1 + \mathbf{o}(A) = \mathbf{o}(1 + A) \leq \mathbf{o}(\mathsf{P_f}(A)) \ .$$

For the upper bound, we will rely on Lemma 2.5.1 to recursively decompose A and prove that $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$.

Theorem 6.2.1. *For all* wqo $(A, \leq)$, $1 + \mathbf{o}(A) \leq \mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$.

Proof. We prove the upper bound induction on $\mathbf{o}(A)$.

If $\mathbf{o}(A)$ is finite. Suppose w.l.o.g. that A is a wpo. Then $\mathbf{o}(A)$ is the cardinal of A and $\mathbf{o}(\mathsf{P_f}(A))$ is less than or equal to the cardinal of $\mathsf{P_f}(A)$ which is $2^{\mathbf{o}(A)}$.

If $\mathbf{o}(A)$ is an infinite successor ordinal, then $\mathbf{o}(A) = \alpha + n$ with α limit and infinite, and $1 \leq n < \omega$. Then using Lemma 2.5.1 we can split $A = A_1 \uplus A_2$ with $\mathbf{o}(A_1) = \alpha$, $\mathbf{o}(A_2) = n$, and $A \geq_{\mathsf{aug}} A_1 \sqcup A_2$.

Combining Lemma 2.1.5 and Equation (Alc-Pf-1), we have $\mathbf{o}(\mathsf{P_f}(A)) \leq \mathbf{o}(\mathsf{P_f}(A_1) \times \mathsf{P_f}(A_2)) = \mathbf{o}(\mathsf{P_f}(A_1)) \otimes \mathbf{o}(\mathsf{P_f}(A_2))$. By induction hypothesis, we know that $\mathbf{o}(\mathsf{P_f}(A_1)) \leq 2^\alpha$, and $\mathbf{o}(\mathsf{P_f}(A_2)) \leq 2^n$. Therefore, $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\alpha+n}$.

If $\mathbf{o}(A)$ is a limit ordinal, then we use Equation (Res-o):

$$\mathbf{o}(\mathsf{P_f}(A)) = \sup_{S \in \mathsf{P_f}(A)} (\mathbf{o}(\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S}) + 1)$$

Let us therefore decompose, given a finite set $S \in \mathsf{P_f}(A)$, the set $\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S}$:

$$\begin{aligned}
\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S} &= \{T \in \mathsf{P_f}(A) \mid S \not\subseteq_{\mathcal{H}} T\} \\
&= \{T \in \mathsf{P_f}(A) \mid \exists x \in S, \forall y \in T, x \not\leq y\} \\
&= \{T \in \mathsf{P_f}(A) \mid \exists x \in S, T \in \mathsf{P_f}(A_{\not\geq x})\} \\
&= \bigcup_{x \in S} \mathsf{P_f}(A_{\not\geq x})
\end{aligned}$$

As a consequence, $\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S}$ is an augmentation of the disjoint union $\bigsqcup_{x \in S} \mathsf{P_f}(A_{\not\geq x})$, and the following inequality holds:

$$\mathbf{o}(\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S}) \leq \bigoplus_{x \in S} \mathbf{o}(\mathsf{P_f}(A_{\not\geq x})) \ .$$

Let us write $\beta = \max_{x \in S} \mathbf{o}(A_{\not\geq x}) < \mathbf{o}(A)$. By induction hypothesis, for all $x \in S$, we have $\mathbf{o}(\mathsf{P_f}(A_{\not\geq x})) \leq 2^{\mathbf{o}(A_{\not\geq x})} \leq 2^\beta$. Since α is limit, 2^α is additively indecomposable. Thus $\mathbf{o}(\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S}) \leq 2^\beta \cdot |S| < 2^\alpha$.

Therefore $\mathbf{o}(\mathsf{P_f}(A)_{\not\supseteq_{\mathcal{H}} S}) + 1 < 2^\alpha$, hence $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^\alpha$. □

Notice that in the proof of Theorem 6.2.1, we related the residual of a set S in $\mathsf{P_f}(A)$ with the residuals of the elements of S in A. This method will be used and reused throughout this chapter.

6.3 . Bounds on the height of $\mathsf{P_f}(A)$

Using the same argument as for $\mathbf{o}(\mathsf{P_f}(A))$, it is clear that the following lower bound holds: $1 + \mathbf{h}(A) \leq \mathbf{h}(\mathsf{P_f}(A))$.

Remark 6.3.1. $\mathsf{P_f}(A)$ is a substructure of the infinitary powerset $\mathsf{P_{inf}}(A)$ ordered with Hoare's embedding, which is isomorphic to $\mathsf{Down}(A)$ the set of downward-closed subsets (also called "initial segments") of A ordered with inclusion. Therefore, $\mathbf{h}(\mathsf{P_f}(A)) \leq \mathbf{h}(\mathsf{Down}(A)) = \mathbf{o}(A) + 1$ (Džamonja et al., 2020, Theorem 3.5).

Theorem 6.3.2. *For all* wqo A*, there exists* $m \in \mathbb{N}$ *such that,*

$$1 + \mathbf{h}(A) \le \mathbf{h}(\mathsf{P_f}(A)) \le \begin{cases} 2^{\mathbf{h}(A)} & \text{when } \mathbf{h}(A) \text{ is a limit ordinal} \\ 2^{\mathbf{h}(A)} \cdot m & \text{when } \mathbf{h}(A) \text{ is a successor ordinal} \end{cases}$$

Proof. We prove the upper bound by induction on $\mathbf{h}(A)$.

If $\mathbf{h}(A)$ is finite then because A is a wqo, A is finite and $\mathsf{P_f}(A)$ is a finite set too. As a consequence, $\mathbf{h}(\mathsf{P_f}(A)) \le 2^{\mathbf{h}(A)+m}$ for some $m \in \mathbb{N}$.

If $\mathbf{h}(A) = \alpha + 1$ then we can leverage Corollary 2.5.3 to decompose $A = A_\perp \uplus A_\top$ such that $\mathbf{h}(A_\perp) = \alpha$ and $A_\top \cong \Gamma_m$ for some $1 \le m < \omega$. Without loss of generality, assume that A is a wpo. Recall that $\mathsf{P_f}(A) \cong \mathsf{Ant}(A)$, the set of antichains of A. We know that $\mathsf{Ant}(A)$ is a wpo, while $\mathsf{P_f}(A)$ might not be. Hence observe that

$$\begin{aligned} \mathsf{Ant}(A) &\le_{\mathsf{aug}} \{ (S_\perp, S_\top) \in \mathsf{Ant}(A_\perp) \cdot \mathsf{Ant}(A_\top) \mid S_\perp \cup S_\top \in \mathsf{Ant}(A) \} \\ &\le_{\mathsf{st}} \mathsf{Ant}(A_\perp) \cdot \mathsf{Ant}(A_\top) \ . \end{aligned}$$

Thus, $\mathbf{h}(\mathsf{Ant}(A)) \le \mathbf{h}(\mathsf{Ant}(A_\perp)) \cdot \mathbf{h}(\mathsf{Ant}(A_\top)) = \mathbf{h}(\mathsf{P_f}(A_\perp)) \cdot \mathbf{h}(\mathsf{P_f}(A_\top))$ according to Lemma 2.1.5 and Table 1.1.

By induction hypothesis, $\mathbf{h}(\mathsf{P_f}(A_\perp)) \le 2^\alpha \cdot k$ for some $k < \omega$ ($k = 1$ if α limit), and $\mathbf{h}(\mathsf{P_f}(\Gamma_m)) = m + 1$ hence $\mathbf{h}(\mathsf{P_f}(A)) \le 2^\alpha \cdot (m + 1)$.

If $\mathbf{h}(A) = \alpha$ where α is limit ordinal then by the Equation (Res-h):

$$\mathbf{h}(\mathsf{P_f}(A)) = \sup_{S \in \mathsf{P_f}(A)} (\mathbf{h}(\mathsf{P_f}(A)_{\sqsubset_{\mathcal{H}} S}) + 1)$$

Let us fix $S \in \mathsf{P_f}(A)$ and let $B \stackrel{\mathsf{def}}{=} \mathsf{P_f}(A)_{\sqsubset_{\mathcal{H}} S}$. Then

$$\begin{aligned} B &= \{T \in \mathsf{P_f}(A) \mid T \sqsubset_{\mathcal{H}} S\} \\ &\subseteq \{T \in \mathsf{P_f}(A) \mid T \subseteq \downarrow S\} \\ &= \mathsf{P_f}(\downarrow S) \end{aligned}$$

We recall that $\downarrow S \stackrel{\mathsf{def}}{=} \bigcup_{x \in S} A_{\le x}$. Because $\mathbf{h}(A) = \alpha$ is a limit ordinal, for all $x \in A$ $\mathbf{h}(A)_{\le x} = \mathbf{h}(A_{<x}) + 1 < \alpha$. Since $S \subseteq A$ is finite, $\mathbf{h}(\downarrow S) < \alpha$.

We can therefore apply the induction hypothesis: There exists $m \in \mathbb{N}$ such that $\mathbf{h}(B) \le 2^{\mathbf{h}(\downarrow S)} < 2^\alpha$ if $\mathbf{h}(B)$ is limit, and $\mathbf{h}(B) \le 2^{\mathbf{h}(\downarrow S)} \cdot m < 2^\alpha$ otherwise.

This proves that $\mathbf{h}(\mathsf{P_f}(A)) \le 2^\alpha$. $\square$

6.4 . Lower bound on the width of $\mathsf{P_f}(A)$

Thanks to Theorem 6.2.1, we can bound the width $\mathbf{w}(\mathsf{P_f}(A))$ by $\mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)}$. This inequality does not bound the width of $\mathsf{P_f}(A)$ by an expression depending on the width of A, and we now claim that there exists no such upper bound.

Example 6.4.1. *For all infinite ordinal α, let $A_\alpha = \omega \sqcup \alpha$. Then $\mathbf{w}(A_\alpha) = 2$ and $\mathbf{w}(\mathsf{P_f}(A_\alpha)) = \alpha$.*

Proof. Since $\mathsf{P_f}(A) \cong \omega \times (1+\alpha)$, we conclude by referring ourselves to Equation (Alc-Pf-1). □

As a consequence, we will focus on providing a lower bound for the width of $\mathsf{P_f}(A)$. As opposed to the case of $\mathbf{h}$ and $\mathbf{o}$, this lower bound happens to be non-trivial to obtain.

Theorem 6.4.2. *Let A be a* wqo*. If $\mathbf{w}(A)$ is infinite then $\mathbf{w}(\mathsf{P_f}(A)) \geq 2^{\mathbf{w}(A)}$. Otherwise $\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(\Gamma_k))$ where $\mathbf{w}(A) = k < \omega$.*

Proof. We prove the result by induction on $\mathbf{w}(A)$.

If $\mathbf{w}(A) = k < \omega$ then one can apply Lemma 2.1.5 and Corollary 2.5.5 to conclude that $\mathbf{w}(\mathsf{P_f}(\Gamma_k)) \leq \mathbf{w}(\mathsf{P_f}(A))$.

If $\mathbf{w}(A) = \alpha + n$ with α limit and $n \geq 1$ then by Lemma 2.5.4, there exists a wqo A' of width α, such that $A' \sqcup \Gamma_n \leq_{\mathsf{st}} A$. Hence $\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(A' \sqcup \Gamma_n)) = \mathbf{w}(\mathsf{P_f}(A') \times \mathsf{P_f}(\Gamma_n))$ according to Lemma 2.1.5 and Equation (Alc-Pf-1).

By induction hypothesis, $2^\alpha \leq \mathbf{w}(\mathsf{P_f}(A'))$. Therefore through Theorem 4.1.3:

$$\begin{aligned}\mathbf{w}(\mathsf{P_f}(A)) &\geq \mathbf{w}(\mathsf{P_f}(A') \times \mathsf{P_f}(\Gamma_n)) \\ &\geq \mathbf{w}(\mathsf{P_f}(A')) \cdot \mathbf{o}(\mathsf{P_f}(\Gamma_n)) \\ &\geq 2^\alpha \cdot 2^n = 2^{\alpha+n} .\end{aligned}$$

If $\mathbf{w}(A) = \omega$ then for all $k < \omega$, $\mathbf{w}(\mathsf{P_f}(A)) \geq \mathbf{w}(\mathsf{P_f}(\Gamma_k))$. Remark that $\sup_{k<\omega} \mathbf{w}(\mathsf{P_f}(\Gamma_k)) = \omega$, and therefore that $\mathbf{w}(\mathsf{P_f}(A)) \geq \omega = 2^\omega$.

If $\mathbf{w}(A) = \alpha > \omega$ and α is a limit ordinal, then for all $\beta < \alpha$, there exists a $A_\beta \leq_{\mathsf{st}} A$ such that $\mathbf{w}(A_\beta) = \beta$ thanks to Lemma 2.1.6. By induction hypothesis, $2^\beta \leq \mathbf{w}(\mathsf{P_f}(A_\beta)) \leq \mathbf{w}(\mathsf{P_f}(A))$.

Therefore, $\mathbf{w}(\mathsf{P_f}(A)) \geq \sup_{\beta<\alpha} 2^\beta = 2^\alpha$. □

Partial conclusion

Let us update Table 1.1 with our newfound results.

Operation	M.O.T.	Height	Width
$A \sqcup B$	$\mathbf{o}(A) \oplus \mathbf{o}(B)$	$\max(\mathbf{h}(A), \mathbf{h}(B))$	$\mathbf{w}(A) \oplus \mathbf{w}(B)$
$A + B$	$\mathbf{o}(A) + \mathbf{o}(B)$	$\mathbf{h}(A) + \mathbf{h}(B)$	$\max(\mathbf{w}(A), \mathbf{w}(B))$
$A \times B$	$\mathbf{o}(A) \otimes \mathbf{o}(B)$	$\mathbf{h}(A) \hat{\oplus} \mathbf{h}(B)$	$\mathbf{o}(A \times B)$ under conditions
$A \cdot B$	$\mathbf{o}(A) \cdot pred_k(\mathbf{o}(B)) + \mathbf{o}(A) \otimes k$ with $k = \mathbf{max\_elt}(B)$	$\mathbf{h}(A) \cdot \mathbf{h}(B)$	$\mathbf{w}(A) \odot \mathbf{w}(B)$
$\mathsf{M}^{\diamond}(A)$	$\omega^{\widehat{\mathbf{o}(A)}}$	$\mathbf{h}^*(A)$	$\omega^{\widehat{\mathbf{o}(A)}-1}$
$\mathsf{M}^{\mathsf{r}}(A)$	$\omega^{\mathbf{o}(A)}$	$\omega^{\mathbf{h}(A)}$	$\omega^{\mathbf{o}_{\perp}(A)}$
$A^*, A \neq \emptyset$	$\omega^{\omega^{(\mathbf{o}(A)^{\pm})}}$	$\mathbf{h}^*(A)$	$\mathbf{o}(A^*)$ if $\mathbf{o}(A) \geq 2$
$\mathcal{T}(A)$	see Schmidt (1979)	$\mathbf{h}^*(A)$	$\mathbf{o}(\mathcal{T}(A))$
$\mathsf{P}_{\mathsf{f}}(A)$	$\leq 2^{\mathbf{o}(A)}$	Theorem 6.3.2	$\geq 2^{\mathbf{w}(A)}$

Table 6.1: Table 1.1 updated.

We computed the missing ordinal invariants when functional (see Theorems 5.2.1 and 5.3.1 for the width of the multiset embedding and the height of the multiset ordering). Otherwise, we managed to recover some functionality by introducing two other invariants: **max_elt** the number of maximal elements of a wqo modulo quasi-isomorphism, which appears in the expression of the mot of the direct product (Theorem 3.0.4), and $\mathbf{o}_{\perp}$ the friendly order type, in which the width of the multiset ordering is functional (Theorem 5.3.3). See how to compute **max_elt** in Proposition 3.0.7, and the friendly order type with Theorems 5.4.4 and 5.4.8. In the cases of the Cartesian product and finite powerset, we found tight bounds (Theorem 4.1.3 for the Cartesian product, Theorems 6.2.1, 6.3.2 and 6.4.2 for the finite powerset) and conditions for when these bounds are reached (Theorem 4.3.2 and Lemma 7.1.1).

Fortunately, everyday-life wqos (such as the ones that appear when studying wsts) are built compositionally from base blocks (ordinals) with the constructions we studied, therefore we can restrict our study to wqos that are part of an elementary family of wqos closed through these constructions, for which we can compute easily all ordinal invariants.

7 - An algebra of well-behaved wqos

We have shown in Section 1.3 that the ordinal invariants of several construction on wqos cannot be expressed as functions in the invariants of smaller wqos. The "misbehaving" operations are the cartesian product and the multiset ordering (their width are not functional) and the finite powerset (on all ordinal invariants). Fortunately we can recover functionality by restricting ourselves to a family of wqos built from ordinals with all the constructions we have seen this far.

Definition 7.0.1 (Elementary wqo)**.** We define the family of *elementary* wqos using the following grammar

$$E \stackrel{\text{def}}{=} \alpha \geq \omega^\omega \text{ mult. indec.} \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid E^* \mid \mathsf{M}^\diamond(E) \mid \mathsf{M}^\mathsf{r}(E) \mid \mathcal{T}(E) \mid \mathsf{P_f}(E)$$

Most elementary wqos can be computed using Table 6.1. However some cases, for instance computing the ordinal invariants of $\mathsf{P_f}(E)$, or computing $\mathbf{w}(E_1 \times E_2)$, need to be handled differently. To reduce the boilerplate code to a minimum, we will first normalise our terms so that we can minimise the appearance of "problematic" computations with the help of some isomorphisms in the spirit of those described in Lemma 2.1.10. The rewriting rules given in Figure 7.1 preserve wqos modulo isomorphism, and define a strongly normalising, confluent rewrite system.

$$\begin{aligned}
\mathsf{P_f}(\alpha) &\to \alpha \;, \\
\mathsf{M}^\mathsf{r}(\alpha) &\to \omega^\alpha \;, \\
E \times (E_1 \sqcup E_2) &\to (E \times E_1) \sqcup (E \times E_2) \;, \\
(E_1 \sqcup E_2) \times E &\to (E_1 \times E) \sqcup (E_2 \times E) \;, \\
\mathsf{M}^\diamond(E_1 \sqcup E_2) &\to \mathsf{M}^\diamond(E_1) \times \mathsf{M}^\diamond(E_2) \;, \\
\mathsf{M}^\mathsf{r}(E_1 \sqcup E_2) &\to \mathsf{M}^\mathsf{r}(E_1) \times \mathsf{M}^\mathsf{r}(E_2) \;, \\
\mathsf{P_f}(E_1 \sqcup E_2) &\to \mathsf{P_f}(E_1) \times \mathsf{P_f}(E_2) \;.
\end{aligned}$$

Figure 7.1: Rewrite rules for elementary wqos

The normal form of a term is computable. We will theferore assume that an elementary wqo is always given via its expression in normal form, i.e., that cannot be rewritten by the above rules.

7.1 . Maximal order type and width of elementary wqos

In this section, we will provide an algorithm to compute both the width and the maximal order type of an elementary wqo E. The main idea of this double computation is that most elementary wqos verify the property $\mathbf{w}(E) = \mathbf{o}(E)$, in which case the following lemma can be applied.

Lemma 7.1.1 (Powerset Sandwich)**.** *Let A be a* wqo *such that $\mathbf{w}(A) = \mathbf{o}(A)$. Then $\mathbf{w}(\mathsf{P_f}(A)) = \mathbf{o}(\mathsf{P_f}(A)) = 2^{\mathbf{o}(A)}$.*

Proof. According to Theorems 6.2.1 and 6.4.2,

$$2^{\mathbf{w}(A)} \leq \mathbf{w}(\mathsf{P_f}(A)) \leq \mathbf{o}(\mathsf{P_f}(A)) \leq 2^{\mathbf{o}(A)} .$$ □

Some elementary wqos do not verify $\mathbf{o}(E) = \mathbf{w}(E)$. Take for instance $E = \alpha$, or E a Cartesian product. Fortunately, if E is a Cartesian product, we can check if $\mathbf{o}(E) = \mathbf{w}(E)$ thanks to Theorem 4.3.2. The same problem arises with the multiset ordering, which means we have to contend with computing a fourth ordinal invariant. Fortunately, we only need to shows that all elementary wqos satisfy Theorem 5.4.8, which gives sufficient conditions for when fot and mot coincide. As for computing the width Cartesian product, it will become clear in Lemma 7.1.2 that most products of elementary wqos satisfy the hypotheses of Theorem 4.3.2.

We now prove that we covered all the necessary cases in the following lemma.

Lemma 7.1.2 (Case disjunction)**.** *Let E be an elementary* wqo *(given through its expression in normal form). Then:*

(i) Either $\mathbf{w}(E) = \mathbf{o}(E) = \mathbf{o}_\perp(E) = \omega^{\omega\cdot\beta}$ for some ordinal $\beta > 0$.
(ii) Or $E = E_1 \sqcup E_2$ with E_1, E_2 elementary wqos*,*
(iii) Or $E = \alpha \geq \omega^\omega$ mult. indec.

Proof. By induction on the expression in normal form of E:

Case $E = \alpha$: then E satisfies (iii).

Case $E = E_1 \sqcup E_2$: then E satisfies (ii).

Case $E = E_1^*$: E satisfies (i), see Table 6.1 and Theorem 5.4.8.

Case $E = \mathsf{M}^\diamond(E_1)$: E is in normal for so by induction hypothesis E_1 satisfies either (i) or (iii). Hence $\mathbf{o}(E_1) > 1$ is additively indecomposable, therefore E satisfies (i), see Table 6.1 and Theorem 5.4.8.

Case $E = \mathcal{T}(E_1)$: E satisfies (i), see Table 6.1 and Theorem 5.4.8.

Case $E = E_1 \times E_2$: E is in normal for so by induction hypothesis E_1 and E_2 both satisfy either (i) or (iii). Thus $\mathbf{o}(E_1)$ and $\mathbf{o}(E_2)$ are of the form $\omega^{\omega\cdot\alpha}$ and $\omega^{\omega\cdot\beta}$ for some $\alpha, \beta > 0$. Therefore according to Theorems 4.3.2 and 5.4.8 $\mathbf{w}(E) = \mathbf{o}(E) = \mathbf{o}(E_1) \otimes \mathbf{o}(E_2) = \omega^{\omega\cdot(\alpha\oplus\beta)}$ and $\mathbf{o}_\perp(E) = \mathbf{o}(E)$, hence E verifies (i).

Case $E = \mathsf{M}^{\mathsf{r}}(E_1)$: E is in normal form so by induction hypothesis E_1 satisfies (i). Then $\mathbf{o}(E) = \omega^{\mathbf{o}(E_1)}$ and $\mathbf{w}(E) = \omega^{\mathbf{o}_\perp(E_1)} = \omega^{\mathbf{o}(E_1)} = \mathbf{o}(E)$. And $\mathbf{o}_\perp(E) = \mathbf{o}(E)$ according to Theorem 5.4.8, hence E verifies (i).

Case $E = \mathsf{P_f}(E_1)$: E is in normal form so by induction hypothesis E_1 satisfies (i). Then $\mathbf{w}(E) = \mathbf{o}_\perp(E) = \mathbf{o}(E) = 2^{\mathbf{o}(E_1)}$ according to Lemma 7.1.1 and Theorem 5.4.8, hence E satisfies (i). □

Proposition 7.1.3 (Observations on the mot, width and fot of elementary wqos)**.** *For any elementary wqo E,*

1. *$\mathbf{o}(E)$ can be written as $\omega^\omega \cdot \alpha$ for some $\alpha > 0$.*

2. *$\mathbf{o}(E)$ is multiplicatively indecomposable iff E is not isomorphic to $E_1 \sqcup E_2$ with E_1, E_2 elementary wqos. If $E \cong E_1 \sqcup E_2$, $\mathbf{o}(E)$ is a sum of multiplicatively indecomposable ordinals.*

3. *$\mathbf{o}_\perp(E) = \mathbf{o}(E)$ iff E is not isomorphic to an ordinal. If E is isomorphic to an ordinal, $\mathbf{o}_\perp(E) = 0$.*

4. *$\mathbf{w}(E) = \mathbf{o}(E)$ iff E is not isomorphic to an ordinal or a disjoint sum of an ordinal and an elementary wqo.*

5. *If E is written in normal form as $\mathsf{P_f}(E_1)$ then $\mathbf{o}(E) = \omega^{\mathbf{o}(E_1)}$.*

Proof. This proof follows directly from the proof of Lemma 7.1.2.

1. True if E satisfies (i) or (iii). If $E = E_1 \sqcup E_2$ then $\mathbf{o}(E) = \mathbf{o}(E_1) \oplus \mathbf{o}(E_2) \overset{\text{(IH)}}{=} \omega^\omega \cdot \alpha \oplus \omega^\omega \cdot \beta = \omega^\omega \cdot (\alpha \oplus \beta)$ for some ordinals $\alpha, \beta > 0$.

2. If E satisfies (i) or (iii), then $\mathbf{o}(E)$ is multiplicatively indecomposable. If $E \cong E_1 \sqcup E_2$, then $\mathbf{o}(E) = \mathbf{o}(E_1) \oplus \mathbf{o}(E_2)$ is not indecomposable (not even additively), but by induction it is a sum of multiplicatively indecomposable ordinals. Interestingly, an elementary wpo is isomorphic to a disjoint sum iff it is a disjoint sum in normal form.

3. Application of Theorem 5.4.8 for the disjoint sum. The fot of a linear wqos is null.

4. For the reverse implication, $\mathbf{w}(\alpha) = 1$ for any ordinal α, and $\mathbf{w}(E_1 \sqcup_2) = \mathbf{w}(E_1) \oplus \mathbf{w}(E_2)$. If $\mathbf{w}(E_i) = \mathbf{o}(E_i)$ for $i \in [1, 2]$, then $\mathbf{w}(E) = \mathbf{o}(E)$.

Otherwise, w.l.o.g. E_1 is isomorphic to an ordinal or a disjoint sum of an ordinal and an elementary wqo by structural induction. Hence E is also isomorphic to a disjoint sum with an ordinal.

5. If $E = \mathsf{P_f}(E_1)$, then $\mathbf{o}(E) = 2^{\mathbf{o}(E_1)}$. If $\alpha = \omega^\omega \cdot \beta$ for some ordinals α, β then $\omega \cdot \alpha = \omega \cdot \omega^\omega \cdot \beta = \omega^\omega \cdot \beta = \alpha$. Hence $2^\alpha = \omega^\alpha$ according to Proposition A.0.17,

□

7.2 . Height of elementary wqos

Given an elementary wqo E, its height $\mathbf{h}(E)$ is recursively defined in terms of the height of its sub-expressions thanks to Table 6.1, with the notable exception of $\mathbf{h}(\mathsf{P_f}(E))$ which cannot be expressed as a function of $\mathbf{h}(E)$.

In this section, we will remove the multiset ordering from the elementary family, for a reason that will be rendered explicit later. Therefore we consider:

Definition 7.2.1 (Elementary$^\diamond$wqo)**.** Here we define the family of *elementary*$^\diamond$ wqos using the following grammar

$$E \stackrel{\mathsf{def}}{=} \alpha \geq \omega^\omega \text{ mult. indec.} \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid E^* \mid \mathsf{M}^\diamond(E) \mid \mathcal{T}(E) \mid \mathsf{P_f}(E)$$

Let us first observe that the height is trivially functional on a variant of elementary wqos where ordinals are limited to ω:

Definition 7.2.2. We define the family of ω-*elementary* wqos using the following grammar:

$$E \stackrel{\mathsf{def}}{=} \omega \mid E_1 \sqcup E_2 \mid E_1 \times E_2 \mid E^* \mid \mathsf{M}^\diamond(E) \mid \mathcal{T}(E) \mid \mathsf{P_f}(E)$$

Proposition 7.2.3. *For any ω-elementary* wqo *E, $\mathbf{h}(E) = \omega$.*

Proof. According to Table 6.1, any ω-elementary wqo which does not contain a finite powerset construction is of height ω. Since $2^\omega = \omega$, Theorem 6.3.2 suffices to conclude. □

Recall that the height of the infinitary powerset construction $\mathbf{h}(\mathsf{P_{inf}}(E))$ can be computed via $\mathbf{h}(\mathsf{P_{inf}}(E)) = \mathbf{o}(E) + 1$ (see Remark 6.3.1). The main contribution of this section, Theorem 7.2.8, derives from this equality to connect the height of the finitary powerset to the supremum of the maximal order type of approximations of E (see Definitions 7.2.4 and 7.2.6).

For any wqo A, for any $n < \omega$, we note $\mathsf{M}^\diamond_n(A)$ the set of multisets of exactly n elements ordered with the multiset embedding. For any $d, b < \omega$

(for depth and branching) we note $\mathcal{T}_{d,b}(A)$ the set of trees of A of depth d such that each internal node has exactly m children, ordered with the tree embedding (i.e. $\mathcal{T}_{0,b}(A) = A$, and $\mathcal{T}_{d,b}(A) \cong A \times (\mathcal{T}^{\times m}_{d-1,b})$). Observe that $\mathcal{T}_{d,b}(A)$ is isomorphic to $A^{\times n}$ where $n = \Sigma_{0 \leq i \leq d} b^i$.

Definition 7.2.4 (Approximation of an elementary$^\diamond$wqo)**.**
Let E be an elementary$^\diamond$wqo.
We say E' *approximates* E, or $E' <_{\mathsf{approx}} E$, iff:

- $E = \alpha$ and $E' \cong \alpha' < \alpha$.
- $E = E_1 \sqcup E_2$ and $E' \cong E'_1 \sqcup E'_2$ where $E'_1 <_{\mathsf{approx}} E_1$ and $E'_2 <_{\mathsf{approx}} E_2$.
- $E = E_1 \times E_2$ and $E' \cong E'_1 \times E'_2$ where $E'_1 <_{\mathsf{approx}} E_1$ and $E'_2 <_{\mathsf{approx}} E_2$.
- $E = E_1^*$ and $E' \cong (E'_1)^{\times n}$ where $E'_1 <_{\mathsf{approx}} E_1$ and $n < \omega$.
- $E = \mathsf{M}^\diamond(E_1)$ and $E' \cong \mathsf{M}^\diamond_n(E'_1)$ where $E'_1 <_{\mathsf{approx}} E_1$ and $n < \omega$.
- $E = \mathcal{T}(E_1)$ and $E' \cong \mathcal{T}_{d,b}(E'_1)$ where $E'_1 <_{\mathsf{approx}} E_1$ and $d, b < \omega$.
- $E = \mathsf{P_f}(E_1)$ and $E' \cong \mathsf{P_f}(E'_1)$ where $E'_1 <_{\mathsf{approx}} E_1$.

The notion of approximation can be understood as a principled way of considering substructures of a given wqo, and the following fact ensures that approximations are indeed substructures.

Fact 7.2.5. $E' <_{\mathsf{approx}} E \implies E' \leq_{\mathsf{st}} E$.

Leveraging, this notion of approximations, we introduce the approximated ordinal invariants.

Definition 7.2.6 (Approximated ordinal invariants)**.** Let $\mathbf{f} \in \{\mathbf{o}, \mathbf{h}, \mathbf{w}\}$, and E be an elementary$^\diamond$wqo. Then, $\underline{\mathbf{f}}(E) \overset{\mathsf{def}}{=} \sup_{E' <_{\mathsf{approx}} E} \{\mathbf{f}(E') + 1\}$.

Fact 7.2.7. *Let E_1, E_2 be elementary$^\diamond$wqos such that the normal form of E_1 is a sub-expression of the normal form of E_2. Then $\underline{\mathbf{f}}(E_1) \leq \underline{\mathbf{f}}(E_2)$.*

We are now ready to state the main technical lemma of this section.

Theorem 7.2.8. *For every elementary$^\diamond$wqo E, $\mathbf{h}(\mathsf{P_f}(E)) = \underline{\mathbf{o}}(E)$.*

E **in normal form**	$\underline{\mathbf{o}}(E)$	**Hypothesis**
$\alpha \geq \omega^\omega$ mult. indec	α	
$E_1 \times E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$	$\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2)$ mult. indec.
$E_1 \sqcup E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$	$\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2)$ mult. indec.
$\mathsf{M}^\diamond(E_1)$	$\underline{\mathbf{o}}(E_1)$	$\underline{\mathbf{o}}(E_1)$ mult. indec.
E_1^*	$\underline{\mathbf{o}}(E_1)$	$\underline{\mathbf{o}}(E_1)$ mult. indec.
$\mathcal{T}(E_1)^*$	$\underline{\mathbf{o}}(E_1)$	$\underline{\mathbf{o}}(E_1)$ mult. indec.
$\mathsf{P_f}(E_1)$	$2^{\underline{\mathbf{o}}(E_1)}$	$\underline{\mathbf{o}}(E_1) = \underline{\mathbf{w}}(E_1)$

Table 7.1: Computing the approximated maximal order type under conditions.

Computing $\underline{\mathbf{o}}(E)$. Before proving Theorem 7.2.8, let us first show how to compute $\underline{\mathbf{o}}(E)$ for any elementary$^\diamond$wqo E, and therefore $\mathbf{h}(\mathsf{P_f}(E))$.

Lemma 7.2.9. *For E an elementary$^\diamond$wqo, $\underline{\mathbf{o}}(E)$ can be computed as described in Table 7.1 in cases following the given hypotheses.*

Proof. Let E_1, E_2 be elementary$^\diamond$wqos, $\alpha = \underline{\mathbf{o}}(E_1)$ and $\beta = \underline{\mathbf{o}}(E_2)$

- Without loss of generality, let $\beta \leq \alpha$. Then, with the hypothesis that α and β are multiplicatively indecomposable:
  - $\underline{\mathbf{o}}(E_1 \sqcup E_2) \leq \sup\{\, \alpha' \oplus \beta' + 1 \mid \alpha' < \alpha, \beta' < \beta \,\} = \alpha$.
  - $\underline{\mathbf{o}}(E_1 \times E_2) \leq \sup\{\, \alpha' \otimes \beta' + 1 \mid \alpha' < \alpha, \beta' < \beta \,\} = \alpha$.
  - $\underline{\mathbf{o}}(E_1^*) \leq \sup\{\, (\alpha')^{\otimes n} + 1 \mid \alpha' < \alpha, n < \omega \,\} \leq \alpha$
  - $\mathsf{M}_n^\diamond(A) \geq_{\mathsf{aug}} (A)^{\times n}$ for any wqo A, hence $\underline{\mathbf{o}}(\mathsf{M}^\diamond(E_1)) \leq \underline{\mathbf{o}}(E_1^*) = \alpha$.
  - $\mathcal{T}_{b,d}(A) \cong (A)^{\times n}$ for some $n < \omega$ arbitrarily large depending on b, d, for any wqo A, hence $\underline{\mathbf{o}}(\mathcal{T}(E_1)) = \underline{\mathbf{o}}(E_1^*) = \alpha$.

  Observe that in these four cases, E_1 is a sub-expression of E, hnce $\underline{\mathbf{o}}(E) \geq \alpha$ according to Fact 7.2.7.

- $2^{\underline{\mathbf{w}}(E_1)} \leq \underline{\mathbf{w}}(\mathsf{P_f}(E_1)) \leq \underline{\mathbf{o}}(\mathsf{P_f}(E_1)) \leq 2^{\underline{\mathbf{o}}(E_1)}$ according to Theorems 6.2.1 and 6.4.2. Hence if $\underline{\mathbf{o}}(E_1) = \underline{\mathbf{w}}(E_1)$, then $\underline{\mathbf{o}}(\mathsf{P_f}(E_1)) = 2^{\underline{\mathbf{o}}(E_1)}$. □

The condition $\underline{\mathbf{o}}(E_1) = \underline{\mathbf{w}}(E_1)$ for $E = \mathsf{P_f}(E_1)$ in Table 7.1 is not very restrictive, since operations like E^* and $\mathsf{M}^\diamond(E)$ do not drastically increase the approximated m.o.t. of an elementary$^\diamond$wqo as they do for the m.o.t., to the point that $\underline{\mathbf{o}} = \underline{\mathbf{w}}$ holds for most elementary$^\diamond$wqos. To prove that, we develop some lower bounds on $\underline{\mathbf{w}}$. While crude, these simple constructions will suffice.

Definition 7.2.10. For any ordinal α, let $P_\alpha = \{ (\beta_0, \beta_1) \mid \beta_0 \leq \beta_1 < \alpha \} \leq_{\mathsf{st}} \alpha^{\times 2}$ the set of increasing pairs of α ordered component-wise.

Observe that $\mathsf{M}^\diamond{}_2(\alpha) \cong P_\alpha$ and $P_\alpha \hookrightarrow \mathsf{P_f}(\alpha \times 2)$ through the function $(\beta_0, \beta_1) \mapsto \{(\beta_1, 0), (\beta_0, 1)\}$.

Lemma 7.2.11. *Let $\alpha \geq \omega^\omega$ be a multiplicatively indecomposable ordinal. Then,*

$$\sup_{\alpha' < \alpha} \mathbf{w}(P_{\alpha'}) \geq \alpha \quad .$$

Proof. Let $\alpha \geq \omega^\omega$, and $\omega \leq \alpha' < \alpha$. Then we can turn any strictly decreasing sequence $x_0, x_1, \ldots$ of $\alpha' - \omega$ into an antichain $(0, x_0 + \omega), (1, x_1 + \omega), \ldots$ of $P_{\alpha'}$ with respect to the prefix order. Hence $\sup_{\omega \leq \alpha' < \alpha} \mathbf{w}(P_{\alpha'}) \geq \sup_{\alpha' < \alpha} \alpha' - \omega = \alpha - \omega = \alpha$. □

As we did with Lemma 7.1.2, let us present a structural lemma which reveals in which cases $\underline{\mathbf{w}} \neq \underline{\mathbf{o}}$.

Lemma 7.2.12 (Case disjunction)**.** *Let E be an elementary$^\diamond$wqo (given through its expression in normal form). Then, $\underline{\mathbf{o}}(E)$ is multiplicatively indecomposable, and one of the following holds:*

(i) $\underline{\mathbf{w}}(E) = \underline{\mathbf{o}}(E)$,

(ii) $E = E_1 \sqcup E_2$ *a disjoint union of elementary$^\diamond$wqos,*

(iii) $E = \alpha$ *an ordinal.*

Proof. Observe that for any elementary$^\diamond$wqo E, $\underline{\mathbf{w}}(E) \leq \underline{\mathbf{o}}(E)$.

By induction on the expression in normal form of E:

Case $E = \alpha$ mult. indec. satisfies (iii). $\underline{\mathbf{o}}(\alpha) = \alpha$ is mult. indec..

Case $E = \mathsf{P_f}(E_1)$: By induction hypothesis E_1 verifies (i) since E is in normal form. Therefore $\underline{\mathbf{w}}(E) = \underline{\mathbf{o}}(E) = 2^{\underline{\mathbf{o}}(E_1)}$. Hence E verifies (i). If $\underline{\mathbf{o}}(E_1)$ is multiplicative indecomposable then $2^{\underline{\mathbf{o}}(E_1)} = \omega^{\underline{\mathbf{o}}(E_1)}$ is multiplicative indecomposable.

Aside from $\mathsf{P_f}$, the approximated m.o.t. of every elementary$^\diamond$operation can be computed under a (multiplicative) indecomposability hypothesis. Moreover Table 7.1 conserves indecomposability.

Case $E = E_1 \sqcup E_2$ satisfies (ii).

Case $E = E_1 \times E_2$: Assume that $\underline{\mathbf{o}}(E_1) = \alpha \geq \beta = \underline{\mathbf{o}}(E_2)$, and therefore that $\underline{\mathbf{o}}(E) = \alpha$ (see Table 7.1). Now there exists $E'_2 <_{\mathsf{approx}} E_2$ such that $\omega \leq_{\mathsf{st}} E'_2$. Thus for any $E'_1 <_{\mathsf{approx}} E_1$, $\mathbf{w}(E'_1 \times E'_2) \geq \mathbf{o}(E'_1)$ according to Corollary 4.1.2. Hence E verifies (i).

Case $E = \mathsf{M}^\diamond(E_1)$: For all $E_1' <_{\mathsf{approx}} E_1$, we know that $\mathsf{M}^\diamond{}_2(E_1') <_{\mathsf{approx}} E$, and $\mathsf{M}^\diamond{}_2(E_1') \leq_{\mathsf{aug}} \mathsf{M}^\diamond{}_2(\mathbf{o}(E_1')) \cong P_{\mathbf{o}(E_1')}$. As a consequence, according to Lemma 7.2.11 and Table 7.1,

$$\underline{\mathbf{w}}(E) \geq \sup_{E_1'} \mathbf{w}(P_{\mathbf{o}(E_1')}) \geq \underline{\mathbf{o}}(E_1) = \underline{\mathbf{o}}(E) \ .$$

Hence E verifies (i).

Case $E = E_1^*$: For any $E_1' <_{\mathsf{approx}} E_1$, for any $n < \omega$, $\mathsf{M}^\diamond_n(E_1') \geq_{\mathsf{aug}} (E_1')^{\times n}$, thus $\underline{\mathbf{w}}(E) \geq \underline{\mathbf{w}}(\mathsf{M}^\diamond(E_1))$. Hence E verifies (i).

Case $E = \mathcal{T}(E_1)$: For any $E_1' <_{\mathsf{approx}} E_1$, for any $b, d < \omega$, $\mathcal{T}_{b,d}(E_1') \cong (E_1')^{\times n}$ for some n arbitrarily large depending on b, d, thus $\underline{\mathbf{w}}(E) = \underline{\mathbf{w}}(E_1^*)$. Hence E verifies (i).

□

Hence, the hypotheses in Table 7.1 are verified for all elementary$^\diamond$wqo E. Therefore, for every elementary$^\diamond$wqo E, one can compute $\mathbf{h}(E)$ with Tables 6.1 and 7.2.

E **in normal form**	$\mathbf{h}(\mathsf{P_f}(E)) = \underline{\mathbf{o}}(E)$
α	α
$E_1 \sqcup E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$
$E_1 \times E_2$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$
$\mathsf{M}^\diamond(E_1)$	$\underline{\mathbf{o}}(E_1)$
E_1^*	$\underline{\mathbf{o}}(E_1)$
$\mathcal{T}(E_1)$	$\underline{\mathbf{o}}(E_1)$
$\mathsf{P_f}(E_1)$	$2^{\underline{\mathbf{o}}(E_1)}$

Table 7.2: Computing the approximated maximal order type inductively.

Proving Theorem 7.2.8. It remains for us to prove the main theorem of this section, that we restate hereafter for readability.

Theorem 7.2.13. *For every elementary$^\diamond$wqo E, $\mathbf{h}(\mathsf{P_f}(E)) = \underline{\mathbf{o}}(E)$.*

The lower bound $\underline{\mathbf{o}}(E) \leq \mathbf{h}(\mathsf{P_f}(E))$ is dealt with through properties of ideal completions (see for instance Halfon (2018)).

Definition 7.2.14 (Ideal Completion)**.** An ideal I of A is a non-empty downwards closed and up-directed subset of A (i.e., for every $x, y \in I$, there exists $z \in I$ such that $x \leq z$ and $y \leq z$).

The set of ideals of A ordered with Hoare's embedding is denoted $\mathsf{Idl}(A)$, and is called the *ideal completion* of A.

Fact 7.2.15 (Goubault-Larrecq et al. (2020))**.** *Let us recall the following facts on ideal completion:*

- *The ideal completion commutes with* $\sqcup$, $\times$, $\mathsf{M}^{\diamond}_{n}$, $\mathcal{T}_{b,d}$ *and* $\mathsf{P_f}$.
- $\mathsf{Idl}(\alpha) = \alpha + 1$ *for any ordinal* α.
- $\mathsf{P_f}(\mathsf{Idl}(A)) \cong \mathsf{P_{inf}}(A)$ *for any* wqo A.

Remark 7.2.16. It is not true that the ideal completion commutes with multiset ordering. Indeed $\mathsf{M^r}(\mathsf{Idl}(\alpha)) \cong \mathsf{M^r}(\alpha+1) \cong \omega^{\alpha+1}$, whereas $\mathsf{Idl}(\mathsf{M^r}(\alpha)) \cong \mathsf{Idl}(\omega^{\alpha}) \cong \omega^{\alpha} + 1$. This is why we removed the multiset ordering from the elementary$^{\diamond}$family in this section.

As a consequence of Fact 7.2.15, the ideal completion of an approximation is an approximation, which we formalise in Fact 7.2.17.

Fact 7.2.17. *Let* $E' <_{\mathsf{approx}} E$ *be two* wqos*. Then* $\mathsf{Idl}(E') <_{\mathsf{approx}} E$.

Lemma 7.2.18. *For every elementary*$^{\diamond}$wqo E, $\underline{\mathbf{o}}(E) \leq \mathbf{h}(\mathsf{P_f}(E))$.

Proof. Let $E' <_{\mathsf{approx}} E$. Then $\mathsf{Idl}(E') <_{\mathsf{approx}} E$ thanks to Fact 7.2.17, and in particular $\mathsf{Idl}(E') \leq_{\mathsf{st}} E$ (Fact 7.2.5), thus $\mathsf{P_f}(\mathsf{Idl}(E')) \leq_{\mathsf{st}} \mathsf{P_f}(E)$. Leveraging Fact 7.2.15 and Lemma 2.1.5, $\mathbf{h}(\mathsf{P_f}(E)) \geq \mathbf{h}(\mathsf{P_f}(\mathsf{Idl}(E'))) = \mathbf{h}(\mathsf{P_{inf}}(E')) = \mathbf{o}(E') + 1$. Hence $\mathbf{h}(\mathsf{P_f}(E)) \geq \underline{\mathbf{o}}(E)$. □

The upper bound $\mathbf{h}(\mathsf{P_f}(E)) \leq \underline{\mathbf{o}}(E)$, in contrast, is quite subtle to prove. Observe that augmentations (Definition 2.1.4) do not preserve the height, depriving us of one of our favorite ways to prove bounds. That is why we introduce the notion of condensation.

Definition 7.2.19 (Condensation)**.** A function $f \colon A \to B$ is a *condensation* if it is surjective, monotonic, and whenever $b \leq_B f(y)$, there exists $x \leq_A y$ such that $b = f(x)$. When there exists a condensation from A to B, we note $B \leq_{\mathsf{cond}} A$.

We have not seen this notion is not known in prior work, as opposed to the standard notions of reflection, augmentation, and substructure. Intuitively, a condensation can be seen as a quotient, where the quotient ordering and the original ordering are related.

Example 7.2.20. *The function* $\iota\colon X^n \to \mathsf{M}^{\diamond}_n(A)$ *mapping* $(x_1, \ldots, x_n)$ *to the multiset* $\langle\, x_1, \ldots, x_n \,\rangle$ *is a condensation.*

Remark 7.2.21. A surjective monotonic function $f\colon A \to B$ is a condensation if and only if the image of a downwards-closed set is itself downwards-closed. In topological terms, f is continuous, closed, and surjective.

The key property of condensations from A to B is that one can simulate decreasing sequences $(b_i)_{i\in\mathbb{N}}$ occurring in B through a careful selection of pre-images $a_i \in f^{-1}(b_i)$. As a consequence, the height of B is controlled by the height of A.

Lemma 7.2.22. *If* $B \leq_{\mathsf{cond}} A$*, then* $\mathbf{h}(B) \leq \mathbf{h}(A)$*.*

Proof. Let $f\colon A \to B$ be a condensation, $b_1 > \cdots > b_{n+1}$ a strictly decreasing sequence of B and $a_1 > \cdots > a_n$ a strictly decreasing sequence of A such that $f(a_i) = b_i$ for all $i \leq n$. Since $b_{n+1} \leq b_n$ there exists $a_{n+1} \leq a_n$ such that $f(a_{n+1}) = b_{n+1}$. Since f is monotonic, $b_{n+1} \not\geq b_n$ implies that $a_{n+1} \not\geq a_n$. Hence $a_1 > \cdots > a_{n+1}$. This shows that to all strictly decreasing sequence of B we can associate a strictly decreasing sequence of A in a way that respects prefix order, hence $\mathsf{Dec}(B)$ is a substructure of $\mathsf{Dec}(A)$ modulo isomorphism. □

Example 7.2.23. *The function* $\iota\colon \mathsf{M}^{\diamond}(A) \to \mathsf{P_f}(A)$ *mapping a multiset* M *to the set* $\{\, x \mid x \in M \,\}$ *is monotonic, surjective, but is not always a condensation.*

Proof. Take $A = 3$. We have $\{0, 1\} \sqsubseteq_{\mathcal{H}} \{2\}$, but a multiset containing at least one 0 and one 1 cannot be dominated by $\{2\}$ no matter the multiplicities. □

We will now introduce two technical tools regarding condensations. The first one is that the constructors used to build approximated wqos are monotonic with respect to $\leq_{\mathsf{cond}}$ (Fact 7.2.24). The second one is that approximations can be extended while respecting $\leq_{\mathsf{cond}}$ (Lemma 7.2.25).

Fact 7.2.24 (Monotonicity)**.** *The operations* $\sqcup$*,* $\times$*,* $\mathsf{P_f}$*, and* $+$ *are monotonic with respect to* $\leq_{\mathsf{cond}}$*.*

Lemma 7.2.25. *Let* E *be an elementary*$^{\diamond}$wqo *and* $E' {<_{\mathsf{approx}}} E$*. Then there exists* $E'' {<_{\mathsf{approx}}} E$ *such that* $1 + E' \leq_{\mathsf{cond+st}} E''$*.*

Proof. By induction on the expression of E in normal form:

Case $E = \alpha$**:** for all $\alpha' < \alpha$, $1 + \alpha' < \alpha$ since α is infinite.

Case $E = E_1 \sqcup E_2$**:** $1 + (E_1' \sqcup E_2') \leq_{\mathsf{cond}} (1 + E_1') \sqcup (1 + E_2')$ for all $E_1' {<_{\mathsf{approx}}} E_1$, $E_2' {<_{\mathsf{approx}}} E_2$.

Case $E = E_1 \times E_2$**:** $1 + (E_1' \times E_2') \leq_{\mathsf{st}} (1 + E_1') \times (1 + E_2')$.

Case $E = E_1^*$**:** $1 + (E_1')^{\times n} \leq_{\mathsf{st}} (1 + E_1')^{\times n}$.

Case $E = \mathsf{M}^\diamond(E_1)$**:** $1 + \mathsf{M}^\diamond_n(E_1') \leq_{\mathsf{st}} \mathsf{M}^\diamond_n(1 + E_1')$.

Case $E = \mathcal{T}(E_1)$**:** $1 + \mathcal{T}_{b,d}(E_1') \leq_{\mathsf{st}} \mathcal{T}_{b,d}\,\mathsf{M}^\diamond_n(1 + E_1')$.

Case $E = \mathsf{P_f}(E_1)$**:** $1 + \mathsf{P_f}(E_1') \leq_{\mathsf{st}} \mathsf{P_f}(1 + E_1')$. □

Lemma 7.2.26. *Let E be an elementary*$^\diamond$wqo*. For all $S \in \mathsf{P_f}(E)$, there exists $E' <_{\mathsf{approx}} E$ such that $\downarrow_E S \leq_{\mathsf{cond+st}} E'$.*

Proof. By induction on the expression in normal form of E:

Case $E = \alpha$**:** Let $\gamma = \max\{\, x \in S \,\}$. Then $\downarrow S \cong \gamma <_{\mathsf{approx}} E$.

Case $E = E_1 \sqcup E_2$**:** $\downarrow_E S \cong \downarrow_{E_1} S_1 \sqcup \downarrow_{E_2} S_2$ where $S_1 = S \cap E_1$ and $S_2 = S \cap E_2$. By induction hypothesis there exist $E_1', E_2' <_{\mathsf{approx}} E_1, E_2$ such that $\downarrow_{E_1} S_1 \leq_{\mathsf{cond+st}} E_1'$, $\downarrow_{E_2} S_2 \leq_{\mathsf{cond+st}} E_2'$. Hence $\downarrow_E S \leq_{\mathsf{cond+st}} E_1' \sqcup E_2'$.

Case $E = E_1 \times E_2$**:** $\downarrow_E S \leq_{\mathsf{st}} \downarrow_{E_1} S_{|E_1} \times \downarrow_{E_2} S_{|E_2}$. By induction hypothesis there exist $E_1', E_2' <_{\mathsf{approx}} E_1, E_2$ such that $\downarrow_{E_1} S_{|E_1} \leq_{\mathsf{cond+st}} E_1'$ and $\downarrow_{E_2} S_{|E_2} \leq_{\mathsf{cond+st}} E_2'$. Hence $\downarrow_E S \leq_{\mathsf{cond+st}} E_1' \times E_2'$.

Case $E = E_1^*$**:** Let n be the maximal length of words in S, and $S' \in \mathsf{P_f}(E_1)$ the set of letters in words of S. Then $\downarrow_E S \leq_{\mathsf{st}} (\downarrow_{E_1} S')^{\leq n}$. Observe that $(\downarrow_{E_1} S')^{\leq n} \leq_{\mathsf{cond}} (1+ \leq_{E_1} S')^{\times n}$: indeed elements of $(1 + \downarrow_{E_1} S')^{\times n}$ can be seen as words in $(\downarrow_{E_1} S')^{\leq n}$ padded with extra bottom elements, the condensation function removing the padding. By induction hypothesis, there exists $E_1' <_{\mathsf{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\mathsf{cond+st}} E_1'$.

Therefore, there exists some $E_1'' <_{\mathsf{approx}} E_1$ given by Lemma 7.2.25, such that $\downarrow_E S \leq_{\mathsf{cond+st}} (1 + E_1')^{\times n} \leq_{\mathsf{cond+st}} (E_1'')^{\times n}$.

Case $E = \mathsf{M}^\diamond(E_1)$**:** Let n be the maximal cardinal of multisets in S, and $S' \in \mathsf{P_f}(E_1)$ the set of elements in multisets of S. Then $\downarrow_E S \leq_{\mathsf{st}} \mathsf{M}^\diamond{}_{\leq n}(\downarrow_{E_1} S') \cong \mathsf{M}^\diamond_n(1 + \downarrow_{E_1} S')$. By induction hypothesis, there exists some $E_1' <_{\mathsf{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\mathsf{cond+st}} E_1'$.

Therefore, there exists some $E_1'' <_{\mathsf{approx}} E_1$ given by Lemma 7.2.25, such that $\downarrow_E S \leq_{\mathsf{cond+st}} \mathsf{M}^\diamond_n(1 + E_1') \leq_{\mathsf{cond+st}} \mathsf{M}^\diamond_n(E_1'')$.

Case $E = \mathcal{T}(E_1)$**:** Let d be the maximal depth of trees in S, and let b be the maximal number of children of nodes of trees of S. Let $S' \in \mathsf{P_f}(E_1)$ be the set of nodes in trees of S. Then $\downarrow_E S \leq_{\mathsf{cond+st}} \mathcal{T}_{b,d}(1 + \downarrow_{E_1} S')$ (same reasoning as in Case $E = E_1^*$). By induction hypothesis, there exists $E_1' <_{\mathsf{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\mathsf{cond+st}} E_1'$. Therefore, there exists some $E_1'' <_{\mathsf{approx}} E_1$ given by Lemma 7.2.25, such that $\downarrow_E S \leq_{\mathsf{cond+st}} \mathcal{T}_{b,d}(1 + E_1') \leq_{\mathsf{cond+st}} \mathcal{T}_{b,d}(E_1'')$.

Case $E = \mathsf{P_f}(E_1)$**:** Let $S' = \{ x \in E_1 \mid \exists y \in S \text{ s.t. } x \in y \} \in \mathsf{P_f}(E_1)$. Then $\downarrow_E S \leq_{\mathsf{st}} \mathsf{P_f}(\downarrow_{E_1} S')$. By induction hypothesis, there exists $E_1' <_{\mathsf{approx}} E_1$ such that $\downarrow_{E_1} S' \leq_{\mathsf{cond+st}} E_1'$, hence $\downarrow_E S \leq_{\mathsf{st}} \mathsf{P_f}(E_1')$. □

Lemma 7.2.27. $\mathbf{h}(\mathsf{P_f}(E)) \leq \underline{\mathbf{o}}(E)$.

Proof. Recall that by Equation (Res-h),

$$\mathbf{h}(\mathsf{P_f}(E)) = \sup_{S \in \mathsf{P_f}(E)} \mathbf{h}(\mathsf{P_f}(E)_{<S}) + 1 \quad .$$

Notice that, given $S \in \mathsf{P_f}(E)$, we have $\mathbf{h}(\mathsf{P_f}(E)_{<S}) + 1 = \mathbf{h}(\mathsf{P_f}(E)_{\leq S})$ and $\mathsf{P_f}(E)_{\leq S} \leq_{\mathsf{st}} \mathsf{P_f}(\downarrow_E S)$.

Using Lemma 7.2.26, there exists $E' <_{\mathsf{approx}} E$ such that $\downarrow_E S \leq_{\mathsf{cond+st}} E'$. As a consequence, $\mathbf{h}(\mathsf{P_f}(E)_{\leq S}) \leq \mathbf{h}(\mathsf{P_f}(E')) \leq \mathbf{h}(\mathsf{P_{inf}}(E')) \leq \mathbf{o}(E') + 1$ according to Fact 7.2.15.

Therefore, for every $S \in \mathsf{P_f}(E)$, $\mathbf{h}(\mathsf{P_f}(E)_{<S} + 1) \leq \underline{\mathbf{o}}(E)$, which implies by Equation (Res-h) that $\mathbf{h}(\mathsf{P_f}(E)) \leq \underline{\mathbf{o}}(E)$. □

Combining Lemmas 7.2.18 and 7.2.27 proves Theorem 7.2.8: $\mathbf{h}(\mathsf{P_f}(E)) = \underline{\mathbf{o}}(E)$.

7.3 . Summary

The two following tables summarize how to compute the ordinal invariants of elementary wqos.

E	$\mathbf{o}(E)$	$\mathbf{w}(E)$	$\mathbf{o}_\perp(E)$
$\alpha \geq \omega^\omega$ mult. indec.	α	1	0
$E_1 \sqcup E_2$	$\mathbf{o}(E_1) \oplus \mathbf{o}(E_2)$	$\mathbf{w}(E_1) \oplus \mathbf{w}(E_2)$	$\mathbf{o}(E)$
$E_1 \times E_2, E_1^*, \mathsf{M}^\diamond(E_1)$ $\mathcal{T}(E_1), \mathsf{M}^\mathsf{r}(E_1)$	Table 6.1	$\mathbf{o}(E)$	$\mathbf{o}(E)$
$\mathsf{P_f}(E_1)$	$2^{\mathbf{o}(E_1)}$	$\mathbf{o}(E)$	$\mathbf{o}(E)$

Table 7.3: How to compute **w** and **o** for elementary wqos in normal form.

E **in normal form**	$\mathbf{h}(E)$	$\underline{\mathbf{o}}(E)$
$\alpha \geq \omega^\omega$ mult.indec	α	α
$E_1 \sqcup E_2$	$\max(\mathbf{h}(E_1), \mathbf{h}(E_2))$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$
$E_1 \times E_2$	$\mathbf{h}(E_1) \hat{\oplus} \mathbf{h}(E_2)$	$\max(\underline{\mathbf{o}}(E_1), \underline{\mathbf{o}}(E_2))$
$\mathsf{M}^\diamond(E_1), E_1^*, \mathcal{T}(E_1)$	$\mathbf{h}^*(E_1)$	$\underline{\mathbf{o}}(E_1)$
$\mathsf{P_f}(E_1)$	$\underline{\mathbf{o}}(E_1)$	$2^{\underline{\mathbf{o}}(E_1)}$

Table 7.4: How to compute **h** and $\underline{\mathbf{o}}$ for elementary$^\diamond$wqos in normal form.

8 - Pushing the limits of the elementary family: when Cartesian product meets direct sum

The direct sum is one of the simplest construction on wqos. Hence the following question: Can we add the direct sum to our elementary family?

The short answer is no. For the finite powerset, Lemma 7.1.1 demands that elementary wqos verify $\mathbf{w}(E) = \mathbf{o}(E)$. For the cartesian product, Theorem 4.3.2 demands that $\mathbf{o}(E)$ be additive indecomposable. However, $\mathbf{o}(A + B) = \mathbf{o}(A) + \mathbf{o}(B)$ whereas $\mathbf{w}(A + B) = (\max \mathbf{w}(A), \mathbf{w}(B))$ (see Table 1.1).

The longer answer is "try at your own risks". Theorem 4.3.2 only hides the complexity of the width of the Cartesian product, we can work without it, but it means refining other tools used in Chapter 4: slices and quasi-incomparable subsets. If you found Chapter 4 quite technical, it gets worse.

Let us study the width of wqos obtained through the following algebra:

$$\mathrm{CP+}\ :\ X \stackrel{\mathsf{def}}{=} \omega^\alpha \geq \omega \mid X_1 \times X_2 \mid X_1 + X_2 \ .$$

As when studying the width of the Cartesian product, we will "slice" a wqo X of CP+ into subsets, and the width of X will be the natural sum of the width of some of these subsets. Let us add more formalism around slicing.

Let S, S' be two subsets of a wqo X. We will use two relations on subsets of a wqo. The first one is the Hoare embedding, that we recall:

$$S \sqsubseteq_{\mathcal{H}} S' \iff \forall x \in S,\ \exists y \in S',\ x \leq_X y \ .$$

The second one is the strict subset ordering:

$$S \prec S' \iff \forall x \in S,\ \forall y \in S',\ x <_X y \ .$$

For all X in CP+, we define an indecomposable slice of X as a subset of X which is isomorphic to a Cartesian product of additively indecomposable ordinals. Let us define inductively a slicing function on CP+, which to a wqo X associates a partition of X into indecomposable slices:

$$\begin{aligned}
\mathsf{Slices}(\omega^\alpha) &\stackrel{\mathsf{def}}{=} \{\omega^\alpha\} \\
\mathsf{Slices}(X_1 \times X_2) &\stackrel{\mathsf{def}}{=} \mathsf{Slices}(X_1) \times \mathsf{Slices}(X_2) \\
\mathsf{Slices}(X_1 + X_2) &\stackrel{\mathsf{def}}{=} \mathsf{Slices}(X_1) \cup \mathsf{Slices}(X_2)
\end{aligned}$$

Let us make a few observations, for X, X_1, X_2 in CP+:

- $\mathsf{Slices}(X)$ is a partition of X into indecomposable slices. Moreover, it is the coarsest partition of X into indecomposable slices.

- $(\mathsf{Slices}(X), \sqsubseteq_{\mathcal{H}})$ is a finite wpo.

- Furthermore $(\mathsf{Slices}(X_1 + X_2), \sqsubseteq_{\mathcal{H}})$ is isomorphic to the direct sum $(\mathsf{Slices}(X_1), \sqsubseteq_{\mathcal{H}}) + (\mathsf{Slices}(X_2), \sqsubseteq_{\mathcal{H}})$ and $(\mathsf{Slices}(X_1 \times X_2), \sqsubseteq_{\mathcal{H}})$ is isomorphic to the Cartesian product $(\mathsf{Slices}(X_1), \sqsubseteq_{\mathcal{H}}) \times (\mathsf{Slices}(X_2), \sqsubseteq_{\mathcal{H}})$.

- From every antichain A in X we can extract an antichain A' in $\mathsf{Slices}(X)$ for the strict subset ordering such that $A \subseteq \bigcup_{S \in A'} S$.

- Any enumeration of slices $S_1, \ldots, S_n$ of $\mathsf{Slices}(X)$ is a family of quasi-incomparable* subsets iff for all $1 \leq i < j \leq n$, $S_i \not\sqsubseteq_{\mathcal{H}} S_j$ and S_i and S_j are incomparable with respect to $\prec$ (generalizing Lemma 4.2.10).

- Every antichain for the strict subset ordering in $\mathsf{Slices}(X)$ can be enumerated as $S_1, \ldots, S_n$ such that for all $1 \leq i < j \leq n$, $S_i \not\sqsubseteq_{\mathcal{H}} S_j$, hence any antichain is, for some enumeration, a family of quasi-incomparable* subsets of X (generalizing Lemma 4.2.11).

From these observations one can deduce the following formula for $\mathbf{w}(X)$.

Lemma 8.0.1. *For all X in CP+,*

$$\mathbf{w}(X) = \max \left\{ \mathbf{w}\left(\bigcup_{S \in A} S \right) \,\middle|\, A \text{ is a } \prec\text{-antichain of } \mathsf{Slices}(X) \right\}.$$

Proof. $(\leq)$ For any $\prec$-antichain A of $Slices(X)$, $\bigcup_{S \in A} S \leq_{\mathsf{st}} X$.

$(\geq)$ The proof of the upper bound follows the same reasoning as in the proof of Lemma 4.1.7: there are a finite number of $\prec$-antichains of $\mathsf{Slices}(X)$. Thus we can "color" every node of $\mathsf{Inco}(X)$ with (one or more) $\prec$-antichain of Slices, such that for all $s \in \mathsf{Inco}(X)$, s is colored with an antichain A iff $s \subseteq X_A \stackrel{\mathsf{def}}{=} \bigcup_{S \in A} S$ and the rank of s in $\mathsf{Inco}(X_A)$ is equal to its rank in $\mathsf{Inco}(X)$. The root of $\mathsf{Inco}(X)$ is colored with at least one antichain A, hence $\mathbf{w}(X) = \mathbf{w}(X_A)$.

□

Since $\bigcup_{S \in A} S \geq_{\mathsf{aug}} \bigsqcup_{S \in A} S$, we have the following upper bound:

$$\mathbf{w}(X) \leq \max \left\{ \bigoplus_{S \in A} \mathbf{w}(S) \,\middle|\, A\ \prec\text{-antichain of } \mathsf{Slices}(X) \right\}. \tag{8.1}$$

On the other hand Lemma 2.3.2 gives us a lower bound:

$$\mathbf{w}(X) \geq \max \left\{ \sum_{i \text{ decreasing}} \mathbf{w}(S_i) \;\middle|\; (S_i) \subseteq \mathsf{Slices}(X) \text{ quasi-incomparable}^\star \right\} .$$

Let us show that the upper bound is not tight with an example.

Lemma 8.0.2. *Let* $X = A \times (B + C)$ *a* wqo*, with* A, B, C *indecomposable ordinals or Cartesian products of indecomposable ordinals, such that* $\mathbf{w}(A \times B) < \mathbf{w}(A \times C)$*. Then* $\mathbf{w}(X) = \mathbf{w}(A \times C)$*.*

Proof. Since $A \times C \leq_{\mathsf{st}} X$, w have the lower bound Then $\mathbf{w}(X) \geq \mathbf{w}(A \times C)$. For the upper bound, let us consider the residuals $X_{\perp(a,b)}$ and $X_{\perp(a,c)}$ for any $a \in A, b \in B, c \in C$.

$$\begin{aligned} X_{\perp(a,b)} &\leq_{\mathsf{st}} (A_{\not\leq a} \times \downarrow_B b) \cup (\downarrow_A a \times B_{\not\leq b}) \cup (\downarrow_A a \times C) \\ &\leq_{\mathsf{st}} (A \times \downarrow_B b) \cup (\downarrow_A a \times B) \cup (\downarrow_A a \times C) \\ &\geq_{\mathsf{aug}} (A \times \downarrow_B b) \sqcup (\downarrow_A a \times B) \sqcup (\downarrow_A a \times C) . \end{aligned}$$

Similarly

$$\begin{aligned} X_{\perp(a,c)} &\leq_{\mathsf{st}} (A_{\not\leq a} \times B) \cup (A_{\not\leq a} \times \downarrow_C c) \cup (\downarrow_A a \times C_{\not\leq c}) \\ &\leq_{\mathsf{st}} (A \times B) \cup (A \times \downarrow_C c) \cup (\downarrow_A a \times C) \\ &\geq_{\mathsf{aug}} (A \times B) \sqcup (A \times \downarrow_C c) \sqcup (\downarrow_A a \times C) . \end{aligned}$$

Now observe that $\mathbf{w}(\downarrow_A a \times C)$, $\mathbf{w}(A \times \downarrow_C c)$ and $\mathbf{w}(A \times B)$ are strictly smaller than $\mathbf{w}(A \times C)$. Furthermore, $\mathbf{w}(A \times \downarrow_B b)$ and $\mathbf{w}(\downarrow_A a \times B)$ are strictly smaller than $\mathbf{w}(A \times B) < \mathbf{w}(A \times C)$. Since $\mathbf{w}(A \times C)$ is multiplicative indecomposable, then $\mathbf{w}(X_{\perp(a,b)})$ and $\mathbf{w}(X_{\perp(a,c)})$ are strictly smaller than $\mathbf{w}(A \times C)$. We conclude with Equation (Res-w).

□

This example can be injected into the general case.

Lemma 8.0.3. *Let* S_1, S_2 *two slices of* X *such that* $S_1 \sqsubseteq_{\mathcal{H}} S_2$ *but* $S_1 \not\prec S_2$*. If* $\mathbf{w}(S_1) < \mathbf{w}(S_2)$ *then* $\mathbf{w}(S_1 \cup S_2) = \mathbf{w}(S_2)$*.*

Proof. W.l.o.g assume that X can be written as a Cartesian product of wqos of CP+. Then, there exists $A, B \in$ CP+, $S' \in \mathsf{Slices}(A)$, $S'_1 \prec S'_2 \in \mathsf{Slices}(B)$, such that $X = A \times B$ and and $S_i = S \times S'_i$ for $i \in \{1, 2\}$. Then $S_1 \cup S_2 \cong S' \times (S'_1 + S'_2)$. Thus according to Lemma 8.0.2, $\mathbf{w}(S_1 \cup S_2) = \mathbf{w}(S_2)$. □

Let us define

$$\mathsf{IncoSl}(X) \stackrel{\mathsf{def}}{=} \left\{ A \text{ antichain of } (\mathsf{Slices}(X), \prec) \;\middle|\; \forall S \sqsubseteq_{\mathcal{H}} S' \in A, \mathbf{w}(S) \geq \mathbf{w}(S') \right\} .$$

For any $A \in \mathsf{IncoSl}(X)$, we define the set of quasi-incomparable enumeration of A:

$$\pi(A) \stackrel{\mathsf{def}}{=} \left\{ S_1, \ldots, S_{|A|} \text{ an enumeration of } A \;\middle|\; \forall i < j, S_i \not\sqsubseteq_{\mathcal{H}} S_j \right\} .$$

Any sequence in $\pi(A)$ is a family of quasi-incomparable subsets of X.

Lemma 8.0.4. *For any* $A \in \mathsf{IncoSl}(X)$*,*

$$\max \left\{ \mathbf{w}(S_{|A|}) + \cdots + \mathbf{w}(S_1) \;\middle|\; S_1, \ldots, S_{|A|} \in \pi(A) \right\} = \bigoplus_{S \in A} \mathbf{w}(S) .$$

Proof. An enumeration $S_1, \ldots, S_{|A|}$ of A satisfies $\mathbf{w}(S_{|A|}) + \cdots + \mathbf{w}(S_1) = \bigoplus_{S\in A} \mathbf{w}(S)$ iff for all $i \leq j$, $\mathbf{w}(S_i) \leq \mathbf{w}(S_j)$. Let us order A by increasing width. In case of equality between two slices $S \sqsubseteq_{\mathcal{H}} S'$, S' should come first. In case of equality between two slices incomparable for $\sqsubseteq_{\mathcal{H}}$, order randomly. This enumeration of A satisfies $\mathbf{w}(S_{|A|}) + \cdots + \mathbf{w}(S_1) = \bigoplus_{S\in A} \mathbf{w}(S)$. It also verifies $\forall i < j, S_i \not\sqsubseteq_{\mathcal{H}} S_j$ by definition of IncoSl, thus it belongs in $\pi(A)$. □

Lemma 8.0.5. *For any antichain A of* $\mathsf{Slices}(X)$ *such that* $A \notin \mathsf{IncoSl}$*, there exists $S_{ind} \in A$ and $A' = A \setminus \{S_{ind}\}$ such that* $\mathbf{w}\left(\bigcup_{S\in A} S\right) = \mathbf{w}\left(\bigcup_{S\in A'} S\right)$*, and*

$$\max_{S_1,\ldots,S_{|A|}\in\pi(A)} \mathbf{w}(S_{|A|}) + \cdots + \mathbf{w}(S_1)$$
$$= \max_{S_1,\ldots,S_{|A|-1}\in\pi(A')} \mathbf{w}(S_{|A|-1}) + \cdots + \mathbf{w}(S_1) .$$

Proof. Since $A \notin \mathsf{IncoSl}$, there exists $S_{\text{ind}} \sqsubseteq_{\mathcal{H}} S' \in A$ such that $\mathbf{w}(S_{\text{ind}}) < \mathbf{w}(S')$. We write A' for $A \setminus \{S_{\text{ind}}\}$. According to Lemma 8.0.3, $\mathbf{w}(S_{\text{ind}} \cup S') = \mathbf{w}(S')$, hence $\mathbf{w}\left(\bigcup_{S\in A} S\right) = \mathbf{w}\left(\bigcup_{S\in A'} S\right)$.

Moreover, let us fix an enumeration $S_1, \ldots, S_{|A|}$ in $\pi(A)$, and let i, j be the indices of S_{ind} and S' in this enumeration. Then $i > j$. Since $\mathbf{w}(S_{\text{ind}}) + \mathbf{w}(S') = \mathbf{w}(S')$, w can conclude. □

Theorem 8.0.6. *For X in CP+,*

$$\mathbf{w}(X) = \max \left\{ \bigoplus_{S\in A} \mathbf{w}(S) \;\middle|\; A \in \mathsf{IncoSl}(X) \right\} .$$

Proof. Lemma 8.0.5 means that all antichains that appear inEquation (8.1) and Chapter 8 are either in $\mathsf{IncoSl}(X)$, or can be replaced by a smaller antichain without loss in the max. We iterate until only antichains of IncoSl remain. Therefore $\mathbf{w}(X) = \max\left\{ \mathbf{w}\left(\bigcup_{S\in A} S\right) \;\middle|\; A \in \mathsf{IncoSl}(X) \right\}$. We conclude with Lemma 8.0.4. □

Conclusion

The main problem that motivated this thesis was to compute the ordinal invariants of some wqos constructions, starting with the ones missing in Džamonja et al. (2020). We found that most of these ordinal invariants have been left unstudied for a good reason: they are not functional in the main three ordinal invariants. Hence our question shifted to: How to deal with non-functional constructions? Can we recover functionality?

Here are some approaches:

1. Finding tight bounds. For most applications of wqos measures, tight upper bounds are quite sufficient. Moreover, bounds can be a stepping stone for further study. For instance, the bounds on the width and mot of the finite powerset provide a sufficient condition for when the width reaches mot.

2. Finding new ordinal invariants. Even if the width of $\mathsf{M}^{\mathsf{r}}(A)$ is not functional in the mot, width, and height of A, it is functional in the friendly order type. Similarly, $\mathbf{o}(A \cdot B)$ becomes functional in $\mathbf{o}(A)$ and $\mathbf{o}(B)$ if you add $\mathbf{max\_elt}(B)$ in the equation. As expected, this approach creates more problems: we now have new ordinal invariants to study.

3. Recovering functionality by restricting ourselves to a family of wqos. For instance, for the width of the Cartesian product, we studied first the Cartesian product of finitely many ordinals. Later, we built an elementary family of wqos for which the ordinal invariants were functional (or at least easily computable for the height).

We used all these approaches to study the missing cases from Table 1.1. See Table 6.1 for a summary of our results. We then showcased how some of our results can be leveraged to measure a large elementary family of wqos.

This work opens many avenues for future research.

Studying the friendly order type. We have introduced a new finite invariant, **max_elt** the number of maximal elements of a wqo, and a new ordinal invariants, $\mathbf{o}_\perp$ the friendly order type (fot). While **max_elt** is quite easy to compute compositionally, fot demands more study. We have already established that fot reaches mot for several wqos constructions (under some conditions). We also gave bounds and tools to compute it. Can we compute the fot compositionally? And does this ordinal invariant have a deeper significance? For instance, does it appear in the computation of other ordinal invariants, or only the width of the multiset ordering? Do open-ended bad sequences appear naturally elsewhere, for instance, controlled open-ended bad sequences in wsts?

Finding new ordinal invariants. Since we can make the mot of the direct product and the width of the multiset ordering functional by inventing new invariants, then maybe similar invariants exist, in which the width of the Cartesian product or the ordinal invariants of the finite powerset would be functional. We mentioned earlier several other wqos measures (the minimal order type, the dimension, the covering numbers) that may be worth studying compositionally.

Adding operations to Table 6.1 and extending the elementary family. Several operations that preserve well quasi-orderedness have not been measured yet. One of the most promising yet daunting wqo constructions is without doubt the minor ordering over finite undirected graphs. Promising, because the minor ordering can be made into a minor embedding by considering edge-labelled graphs. As we have seen studying Table 1.1, the ordinal invariants of embedding orders, like the finite multiset embedding, the finite words embedding, and the finite tree homeomorphic embedding all fit a clear pattern: height in $\mathbf{h}^*(A)$ and width reaches mot, which is functional in $\mathbf{o}(A)$ and multiplicative indecomposable (under conditions). All these properties make embeddings easy to integrate into the elementary family. Hopefully, the graph minor embedding also respects this pattern. Daunting, because the proof of the Robertson-Seymour theorem is developed through no less than twenty articles. Computing the mot of the graph ordering would be an alternative proof of the Robertson-Seymour theorem. One can expect this mot to be larger than the mot of $\mathcal{T}(A)$, thus requiring notations systems for bigger ordinals, such as Schütte's Klammersymbols. One small ray of hope: the proof of the Robertson-Seymour Theorem starts this way: for any $G \in \mathcal{G}(A)$, consider the residual $\mathcal{G}(A)_{\not\geq G}$ and observe that any graph in this residual has a sort of tree decomposition. This hints at the method of residual reducting to trees.

Noetherian spaces. Noetherian spaces are a topological generalisation of wqos (every wqo is a Noetherian space in its Alexandroff topology). The stature of a Noetherian space is the equivalent of the maximal order type of a wqo (the name is inspired from Blass and Gurevich (2008) who named their maximal order type stature). Goubault-Larrecq and Laboureix (2023) showed that the mot of a wqo is equal to its stature when equipped with the Alexandroff topology. Another notion of measure on Noetherian spaces is the sobrification rank, which does not correspond to the width nor the height. These two ordinal invariants can be computed compositionally, see for instance Goubault-Larrecq et al. (2023) for the stature and sobrification rank of transfinite words with the regular subword topology. Thus we can study wqos constructions transposed to Noetherian spaces. Another intriguing question would be: is the sobrification rank of a wqo (equipped with the Alexandroff topology) a relevant ordinal invariant to study ?

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A - Ordinal arithmetic

Several proofs in this thesis rely on tedious computations on wqo invariants, i.e., on ordinals. Hopefully the reader is comfortable with the usual ordinal operations: the ordinal sum, product and exponentiation, the natural sum and product. In this appendix, we will recall their definition and properties (see any textbook like Holz et al. (1999) for a fully detailed introduction). We will also tackle less-known operations, such as the Jacobsthal product, or the left subtraction.

Normal forms.

Definition A.0.1 (Cantor normal form)**.** Any ordinal α can be expressed in *Cantor normal form*, or CNF, as $\alpha = \sum_{i<n} \omega^{\alpha_i}$, where $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1}$ are the *exponents* of α. This expression is unique.

Definition A.0.2 (Cantor normal form with coefficient)**.** Any ordinal α can be expressed in *CNF with coefficient* as $\alpha = \sum_{i<n} \omega^{\alpha_i} \cdot a_i$, where $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1}$ are the *exponents* of α, and $0 < a_i < \omega$ for $i < n$. This expression is unique.

Any ordinal α can also be expressed uniquely as $\omega \cdot \alpha' + k$ with $k < \omega$, by left division by ω. We call k the finite part of α, and $\omega \cdot \alpha'$ its limit part.

In this chapter, we will give algorithmic intuitions about ordinal operations where ordinal are given and handled either through their Cantor normal form or their limite-finite decomposition.

Ordinal sum and product.

Definition A.0.3 (Ordinal sum)**.** $\alpha + \beta$ is defined inductively on β.

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \lambda &= \sup_{\gamma<\lambda}(\alpha + \gamma),\ \lambda \text{ limit} \end{aligned}$$

Definition A.0.4 (Ordinal product)**.** $\alpha \cdot \beta$ is defined inductively on β.

$$\begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \lambda &= \sup_{\gamma<\lambda} \alpha \cdot \gamma,\ \lambda \text{ limit} \end{aligned}$$

The ordinal addition and multiplication are not commutative : $1+\omega = \omega \neq \omega+1$, $2\cdot\omega = \omega \neq \omega\cdot 2$. That aside, they have many nice properties : they are associative, continuous and strictly increasing in the right argument, while just increasing in the left one:

$$\alpha+\beta < \alpha+\beta' \text{ and } \beta+\alpha \leq \beta'+\alpha \text{ when } \beta < \beta' \,. \tag{A.1}$$

The multiplication is left-distributive over the addition:

$$\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma \,. \tag{A.2}$$

Definition A.0.5 (Indecomposable ordinals (Holz et al., 1999))**.** An ordinal α is

- *additively indecomposable* (add. indec.) when for any $\beta < \alpha$ and $\gamma < \alpha$, $\beta+\gamma < \alpha$. Additively indecomposable ordinals can also be characterized as ordinals the form $\omega^{\alpha'}$ with α' any ordinal.

- *multiplicatively indecomposable* (mult. indec.) when for any $\beta < \alpha$ and $\gamma < \alpha$, $\beta\cdot\gamma < \alpha$. Multiplicatively indecomposable ordinals (besides 2) can also be characterized as ordinals of the form $\omega^{\alpha'}$ with α' additively indecomposable.

Remark A.0.6. In this thesis, we chose not to consider 0, 1 and 2 as indecomposable, neither additively nor multiplicatively. Thus when we write indecomposable ordinal, we mean implicitly infinite indecomposable ordinals.

Lemma A.0.7 (Ordinal sum: CNF characterization (Manolios and Vroon, 2005))**.** *Let $\alpha = \sum_{i<n}\omega^{\alpha_i}$ and $\beta = \sum_{i<m}\omega^{\beta_i}$ in CNF. Then*

$$\alpha+\beta = \sum_{i<k}\omega^{\alpha_i} + \sum_{i<m}\omega^{\beta_i} \text{ where } k = \min\{\, i<n \mid \alpha_i < \beta_0 \,\} \cup \{n\} \,. \tag{A.3}$$

Equation (A.3) (and later Equations (A.4) and (A.5)) are proven in Manolios and Vroon (2005) for $\alpha, \beta < \epsilon_0$. This condition makes sense in the article, which aims to implement ordinal operations in a proof assistant, for a representation of ordinals up to ϵ_0. However, the proofs do not use this condition, hence these results can be extended to any ordinals.

Lemma A.0.8 (Ordinal product: CNF characterization (Manolios and Vroon, 2005))**.** *Let $\beta = \sum_{i<m}\omega^{\beta_i}$ in CNF be a limit ordinal (i.e. $\beta_{m-1} > 0$). Let $k < \omega$. Let $\alpha = \omega^{\alpha_0}\cdot a+\sigma$ with $\sigma < \omega^{\alpha_0}$, $a < \omega$. Then*

$$\alpha\cdot(\beta+k) = \begin{cases} \sum_{i<m}\omega^{\alpha_0+\beta_i} & \text{if } k=0, \\ (\sum_{i<m}\omega^{\alpha_0+\beta_i}) + \omega^{\alpha_0}(a\cdot k)+\sigma & \text{otherwise.} \end{cases} \tag{A.4}$$

Left subtraction. If $\beta \leq \alpha$, then the subtraction $\alpha - \beta$ is the unique ordinal γ such that $\beta + \gamma = \alpha$. Left division is defined similarly, but we will never use it.

Lemma A.0.9 (Left subtraction: CNF characterization (Manolios and Vroon, 2005))**.** *Let* $\alpha = \sum_{i<n} \omega^{\alpha_i}$ *and* $\beta = \sum_{i<m} \omega^{\beta_i}$ *in CNF. Then*

$$\alpha - \beta = \sum_{k \leq i < n} \omega^{\alpha_i} \textit{ where } k = \min \{ \, i < n \mid \alpha_i \neq \beta_i \, \} \cup \{m, n\} \, . \tag{A.5}$$

Proof. Let $k = \min \{ \, i < n \mid \alpha_i \neq \beta_i \, \} \cup \{m, n\}$. If $\alpha = \beta$ then $k = n$ and $\alpha - \beta = 0$.

Otherwise let $\alpha > \beta$. Therefore if $k < m$ then $\alpha_k > \beta_k$.

$$\begin{aligned}
\beta + \sum_{k \leq i < n} \omega^{\alpha_i} &= \sum_{i<m} \omega^{\beta_i} + \sum_{k \leq i < n} \omega^{\alpha_i} \\
&= \sum_{i<k} \omega^{\alpha_i} + (\sum_{k \leq i < m} \omega^{\beta_i} + \sum_{k \leq i < n} \omega^{\alpha_i}) \\
&= \sum_{i<k} \omega^{\alpha_i} + \sum_{k \leq i < n} \omega^{\alpha_i} = \alpha \text{ by Equation (A.3).}
\end{aligned}$$

□

Left subtraction is strictly increasing in the left argument:

$$\alpha - \beta < \alpha' - \beta \text{ when } \beta \leq \alpha < \alpha' \, . \tag{A.6}$$

Observe that subtracting does not always mean decreasing. For $\alpha = \sum_{i<n} \omega^{\alpha_i}$ and $\beta = \sum_{i<m} \omega^{\beta_i}$ in CNF,

$$\alpha - \beta = \alpha \text{ iff } \alpha_0 > \beta_0 \tag{A.7}$$

In particular, $\alpha - 1 = \alpha$ iff α is infinite. Furthermore, if α is additively indecomposable and $\beta < \alpha$, then $\alpha - \beta = \alpha$.

Left subtraction is counter intuitive: $(\omega + 1) - 1 = \omega + 1$ and not ω. Nonetheless we might sometimes need a predecessor operation:

$$\text{pred}(\alpha) = \sup \{ \, \beta < \alpha \, \} \, .$$

Thus defined, $\text{pred}(\beta + 1) = \beta$ for any ordinal β, but the predecessor operation leaves limit ordinals unchanged.

Natural operations. The natural sum and product have several equivalent definitions. The simplest one is obtained by treating the Cantor normal form as a polynomial, and define the sum and product accordingly.

Definition A.0.10 (Natural sum and product)**.** For any ordinals α, β with $\alpha = \sum_{i<n} \omega^{\alpha_i}$ and $\beta = \sum_{i<m} \omega^{\beta_i}$ in CNF,

- The *natural sum*, or Hessenberg sum, $\alpha \oplus \beta$ is $\gamma = \sum_{i<n+m} \omega^{\gamma_i}$ with $\gamma_0 \geq \cdots \geq \gamma_{n+m-1}$ being a reordering of $\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{m-1}$.

- The *natural product*, or Hessenberg product, $\alpha \otimes \beta$ is $\bigoplus_{i<n,j<m} \omega^{\alpha_i \oplus \beta_j}$.

Another characterization is that the natural sum (resp. product) of two ordinals is the mot of the disjoint sum (resp. Cartesian product) of these ordinals (de Jongh and Parikh, 1977).

Both natural operations are commutative, associative and strictly increasing in both arguments.

We always have the inequalities $\alpha + \beta \leq \alpha \oplus \beta$ and $\alpha \cdot \beta \leq \alpha \otimes \beta$. These inequalities are tight. For $\alpha = \sum_{i<n} \omega^{\alpha_i}$ and $\beta = \sum_{i<m} \omega^{\beta_i}$ in CNF

$$\alpha + \beta = \alpha \oplus \beta \text{ iff } \alpha_{n-1} \geq \beta_0 \ . \tag{A.8}$$

In particular, $\alpha + n = \alpha \oplus n$ for any $n < \omega$. On the other hand,

$$\alpha + \beta = \beta \text{ iff } \beta_0 > \alpha_0 \ . \tag{A.9}$$

In particular, $1 + \alpha = \alpha$ iff α is infinite.

It is harder to express necessary conditions for coincidence of the ordinal and natural products, but we can give a sufficient one:

Lemma A.0.11. *If α is additively indecomposable and $\beta < \omega^\omega$, then $\alpha \cdot \beta = \alpha \otimes \beta$.*

Proof. Let $\alpha = \omega^{\alpha_0}$ and $\beta = \sum_{i<m} \omega^{\beta_i}$ such that $\beta_i < \omega$ for all $i < m$. Then $\alpha \cdot \beta \overset{A.4}{=} \sum_{i<m} \omega^{\alpha_0 + \beta_i} \overset{A.8}{=} \sum_{i<m} \omega^{\alpha_0 \oplus \beta_i} = \alpha \otimes \beta$. □

Lemma A.0.12. *If $\alpha \leq \beta \oplus \gamma$ and α is additively indecomposable then either $\alpha \leq \beta$ or $\alpha \leq \gamma$.*

Proof. By contraposition, if $\beta < \alpha$ and $\gamma < \alpha$ then $\gamma \oplus \beta = \sum_{i<n+m} \omega^{\gamma_i}$ for some ordinals γ_i for all $i < n + m$, such that $\omega^{\gamma_i} < \alpha$. Hence by definition of indecomposability, $\gamma \oplus \beta < \alpha$. □

Lemma A.0.13. *For any ordinals α, β, γ, if γ is of the form $\gamma = \omega^{\gamma_0} \cdot l$ with $l < \omega$ then $(\alpha + \beta) \otimes \gamma = \alpha \otimes \gamma + \beta \otimes \gamma$.*

Proof. Let $\alpha = \sum_{i<n} \omega^{\alpha_i}$ and $\beta = \sum_{i<m} \omega^{\beta_i}$ in CNF, and $k = \min\{\, i < n \mid \alpha_i < \beta_0 \,\} \cup \{n\}$. Then

$$
\begin{aligned}
\text{Then } (\alpha+\beta)\otimes\gamma &= (\sum_{i<k} \omega^{\alpha_i} + \sum_{i<m} \omega^{\beta_i}) \otimes \omega^{\gamma_0} \cdot l \text{ (Equation (A.3))}\\
&= (\bigoplus_{i<k} \omega^{\alpha_i} \oplus \bigoplus_{i<m} \omega^{\beta_i}) \otimes \omega^{\gamma_0} \cdot l \text{ (Equation (A.8))}\\
&= \bigoplus_{i<k} \omega^{\alpha_i \oplus \gamma_0} \cdot l \oplus \bigoplus_{i<m} \omega^{\beta_i \oplus \gamma_0} \cdot l\\
\text{whereas } \alpha\otimes\gamma + \beta\otimes\gamma &= (\bigoplus_{i<n} \omega^{\alpha_i}) \otimes \gamma + (\bigoplus_{i<m} \omega^{\beta_i}) \otimes \gamma \text{ (Equation (A.8))}\\
&= \bigoplus_{i<n} \omega^{\alpha_i + \gamma_0} \cdot l + \bigoplus_{i<m} \omega^{\beta_i + \gamma_0} \cdot l\\
&= \bigoplus_{i<k} \omega^{\alpha_i \oplus \gamma_0} \cdot l \oplus \bigoplus_{i<m} \omega^{\beta_i \oplus \gamma_0} \cdot l \text{ (Equation (A.3))}
\end{aligned}
$$

□

Jacobsthal product. However, one can derive another multiplication from the natural sum, the Jacobsthal product (Jacobsthal, 1909). This product was rediscovered by Abraham and Bonnet (1999) where it is called the Hessenberg-based product.

Definition A.0.14 (Jacobsthal product)**.** The Jacobsthal product $\alpha \odot \beta$ is defined inductively on β.

$$
\begin{aligned}
\alpha \odot 0 &= 0\\
\alpha \odot (\beta+1) &= (\alpha \odot \beta) \oplus \alpha\\
\alpha \odot \lambda &= \sup_{\gamma<\lambda} \alpha \odot \gamma,\ \lambda \text{ limit}
\end{aligned}
$$

We always have $\alpha \cdot \beta \leq \alpha \odot \beta \leq \alpha \otimes \beta$. These inequalities can be strict. Take for instance $\alpha = \beta = \omega + 2$. Then $\alpha \cdot \beta = \omega^2 + \omega \cdot 3 + 2$, $\alpha \odot \beta = \omega^2 + \omega \cdot 4 + 2$, $\alpha \otimes \beta = \omega^2 + \omega \cdot 4 + 4$.

The Jacobsthal product does not differ much from the ordinal multiplication:

Lemma A.0.15 (Jacobsthal product: CNF characterization)**.** *Let $\alpha = \omega^{\alpha_0} \cdot a + \sigma$ with $a < \omega$ and $\sigma < \omega^{\alpha_0}$. Let β be a limit ordinal and $k < \omega$.*

$$
\begin{aligned}
\alpha \odot (\beta + k) &= \alpha \cdot \beta + \alpha \otimes k\\
&= \omega^{\alpha_0} \cdot \beta + \omega^{\alpha_0} \cdot ak + (\sigma \otimes k)
\end{aligned}
$$

Proof. By induction on $\beta + k$. By definition, $\alpha \odot (\beta + k) = \alpha \odot \beta \oplus \alpha \otimes k$. If $\beta = 0$ it is easy. Otherwise $\alpha \odot \beta = \sup_{\beta' < \beta} \alpha \odot \beta'$. Any $\beta' < \beta$ can be written as $\beta' = \lambda + k'$ with λ limit, $k' < \omega$. Then by induction hypothesis $\alpha \odot \beta' = \alpha \cdot \lambda + \alpha \otimes k' < \alpha \cdot \lambda + \alpha \cdot (k+1) < \alpha \cdot \beta$ according to Equation (A.4). □

Exponentiation As there are several products derived from the ordinal and natural sums, there are several exponentiations derived from those products (Altman (2017)). The only one we will use is the usual exponentiation:

Definition A.0.16 (Ordinal exponentiation)**.** α^β is defined inductively on β.

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \cdot \alpha \\ \alpha^\lambda &= \sup_{\gamma<\lambda} \alpha^\gamma, \ \lambda \text{ limit} \end{aligned}$$

Furthermore, we actually only use $\alpha \mapsto \omega^\alpha$ and $\alpha \mapsto 2^\alpha$.

In Chapter 6, we often use the ordinal exponentiation of 2. The following Proposition is all you need to know about exponentiation to understand these thesis.

Proposition A.0.17 (Ordinal exponentiation of 2Manolios and Vroon (2005).)**.** *Let $\alpha = \omega \cdot \alpha' + n$ with $n < \omega$. Then $2^\alpha = \omega^{\alpha'} \cdot 2^n$. Hence, if α is limit then 2^α is additive indecomposable. If α is additive indecomposable, then 2^α is multiplicatively indecomposable.*

Proof. $2^\alpha = 2^{\omega\cdot\alpha'+n} = (2^\omega)^{\alpha'} \cdot 2^n$ and $2^\omega = \sup_{n<\omega} 2^n = \omega$. □