The true colors of memory: A tour of chromatic-memory strategies in zero-sum games on graphs

Patricia Bouyer

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France

Line of works developed together with Mickael Randour and Pierre Vandenhove.
Some works are co-authored with other people: Antonio Casares, Nathanaël Fijalkow, Stéphane Le Roux, Youssouf Oualhadj.

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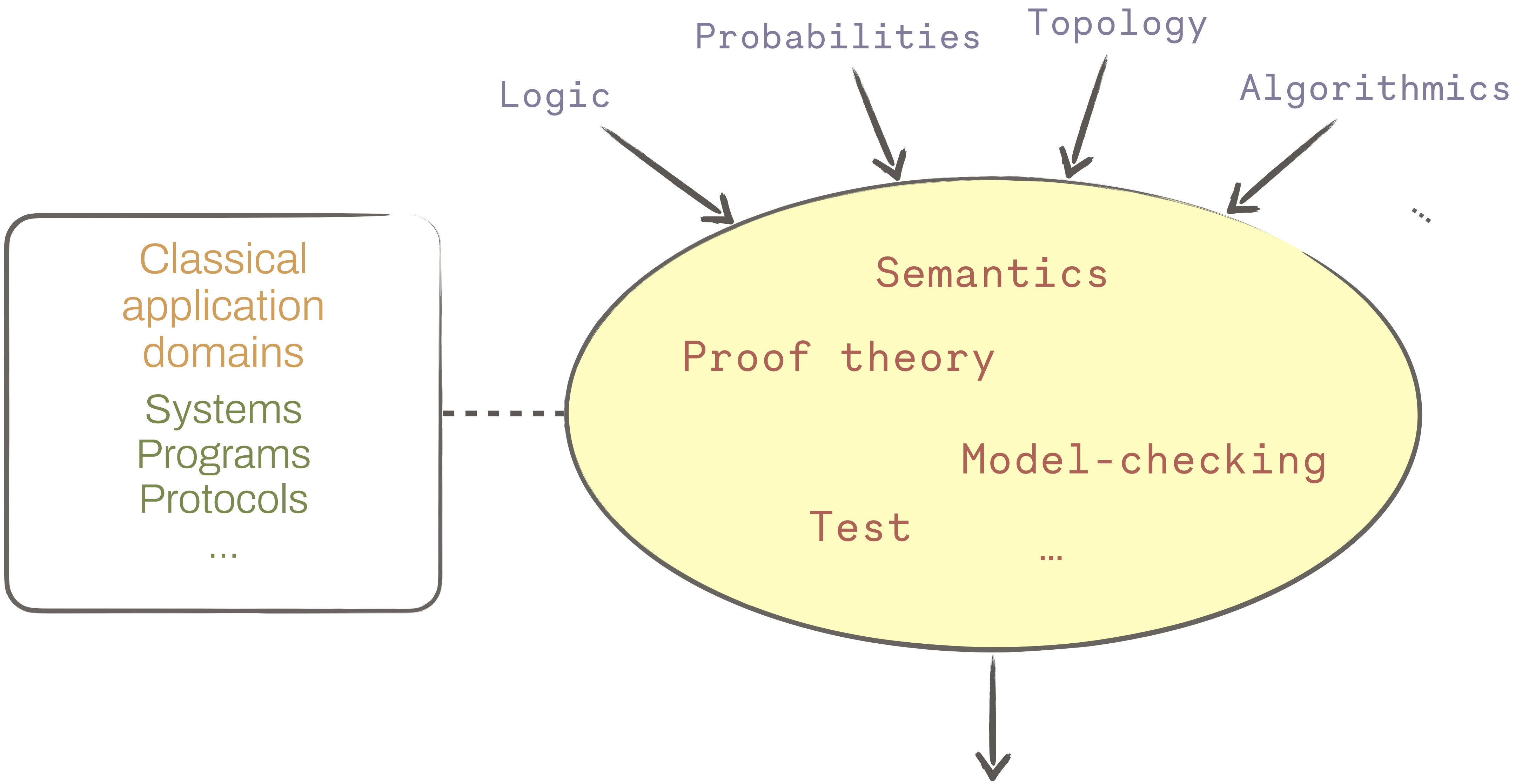
Motivation

—

The setting

My field of research: Formal methods



Logic

Probabilities

Topology

Algorithmics

…

Classical application domains

Systems
Programs
Protocols
…

Semantics

Proof theory

Model-checking

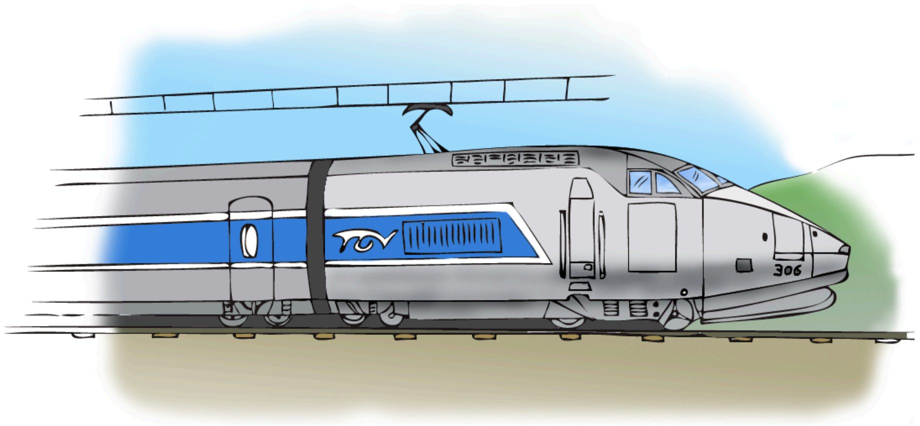
Test

…

Give guarantees (+ certificates) on functionalities or performances

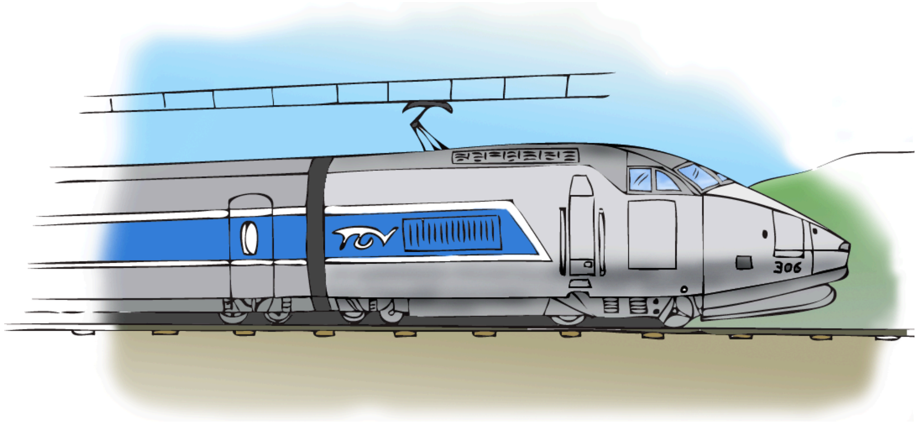
Model-checking

System



Model-checking

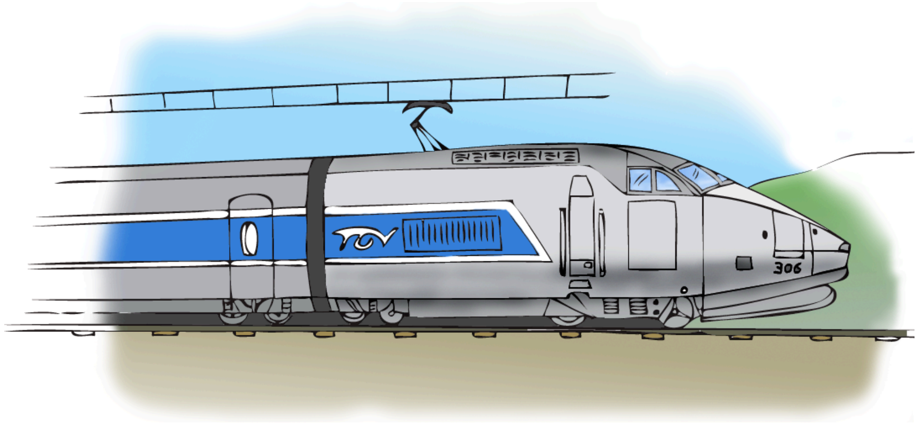
System



Properties

Model-checking

System



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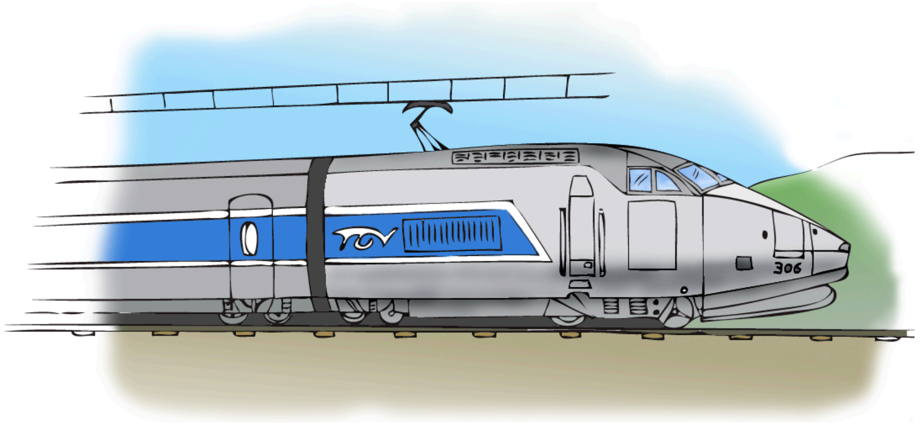

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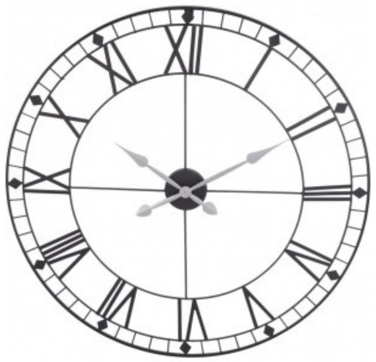

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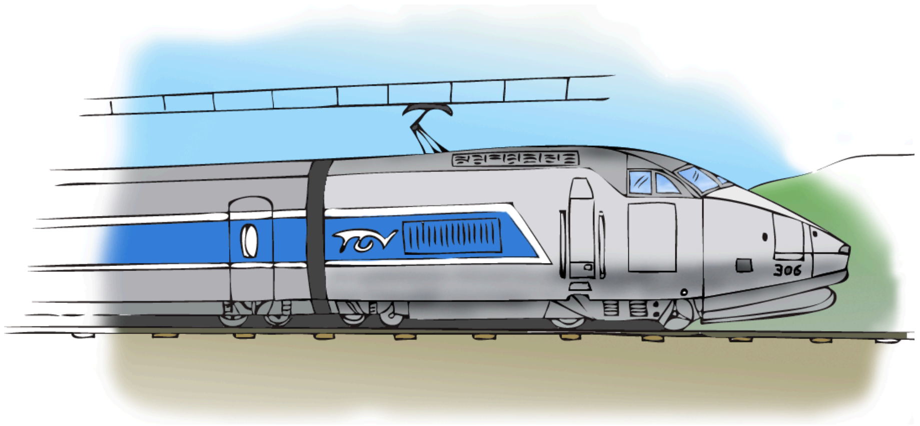

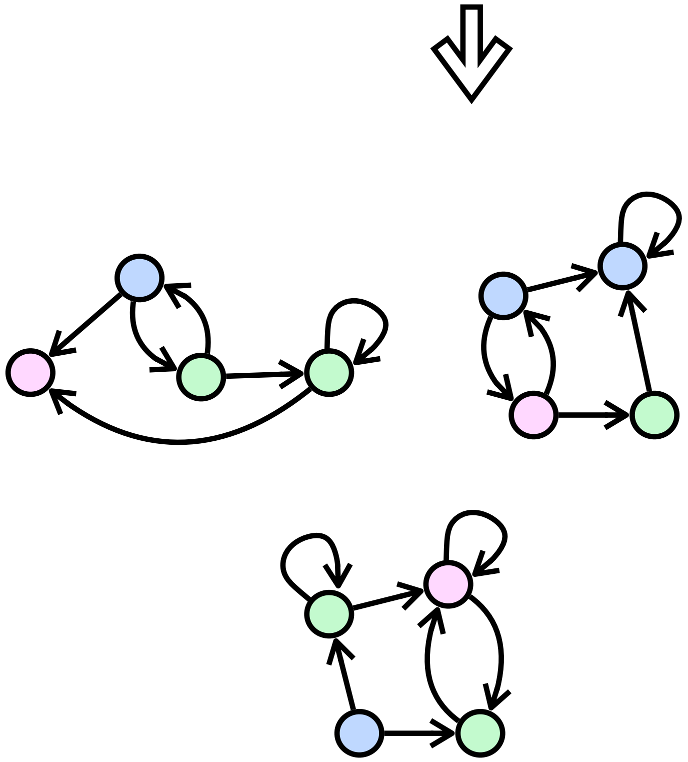

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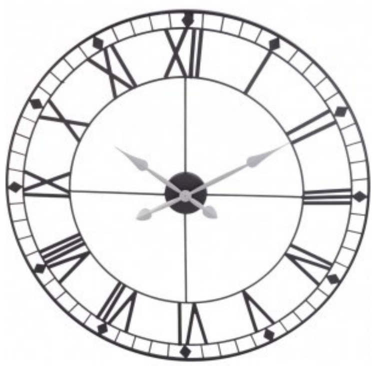

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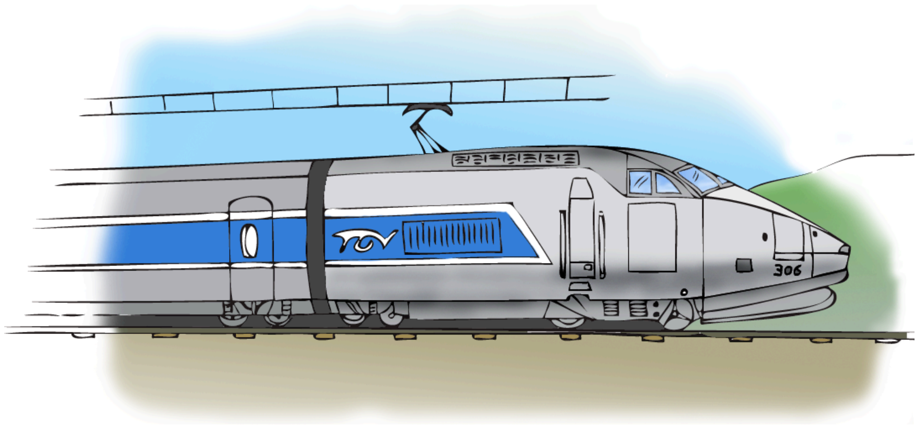
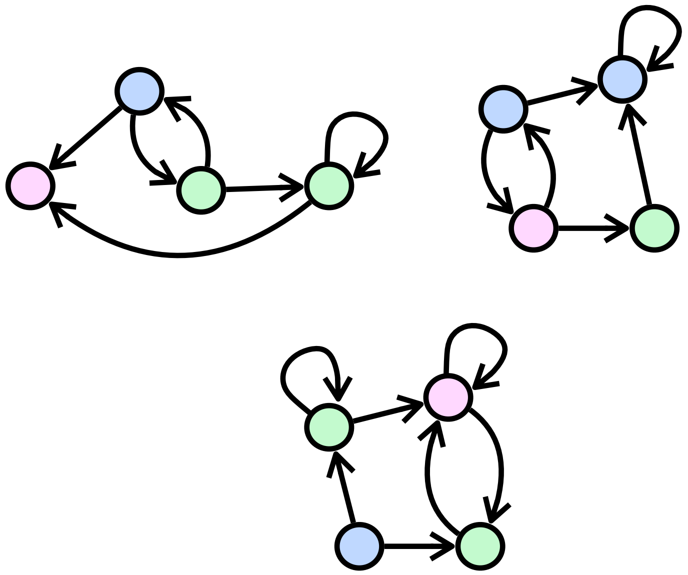


⇩



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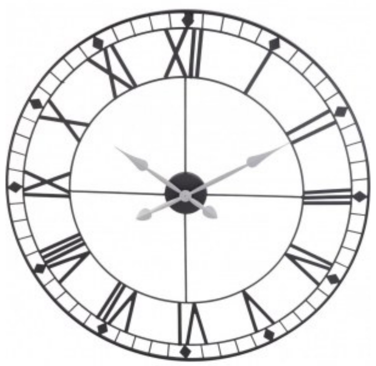
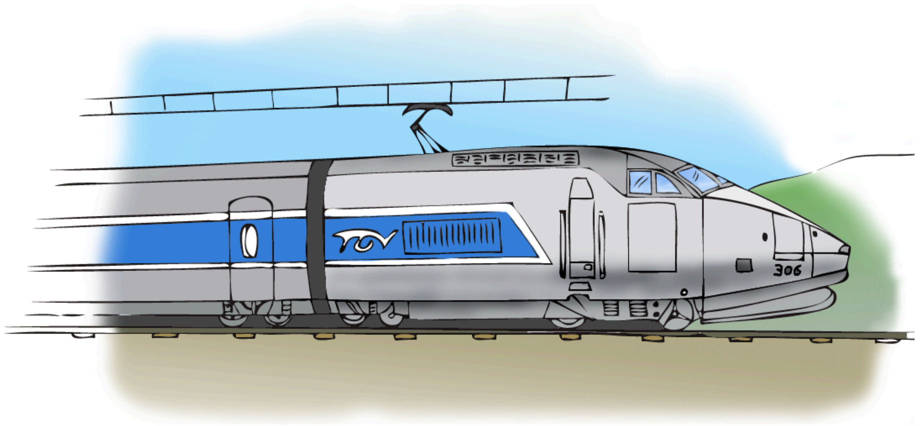


⇩

$$\varphi = \mathbf{AG}\,\neg\text{crash} \wedge \left(\mathbb{P}(\mathbf{F}_{\leq 2\text{h}}\text{arr}) \geq 0{,}9\right)$$

Model-checking

System



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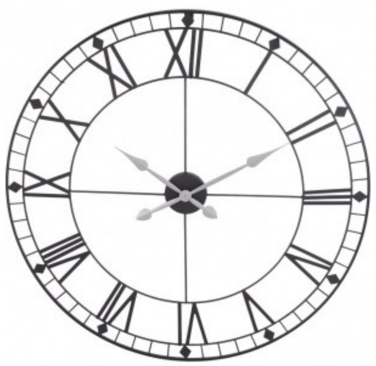

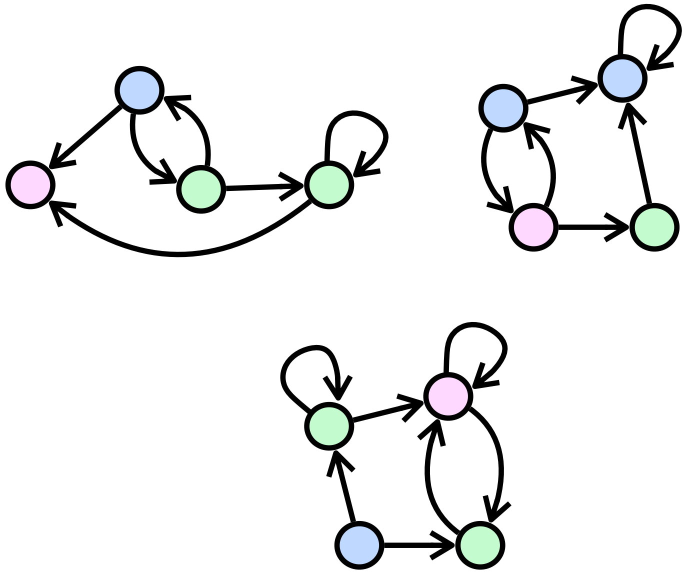

Model-checking algorithm

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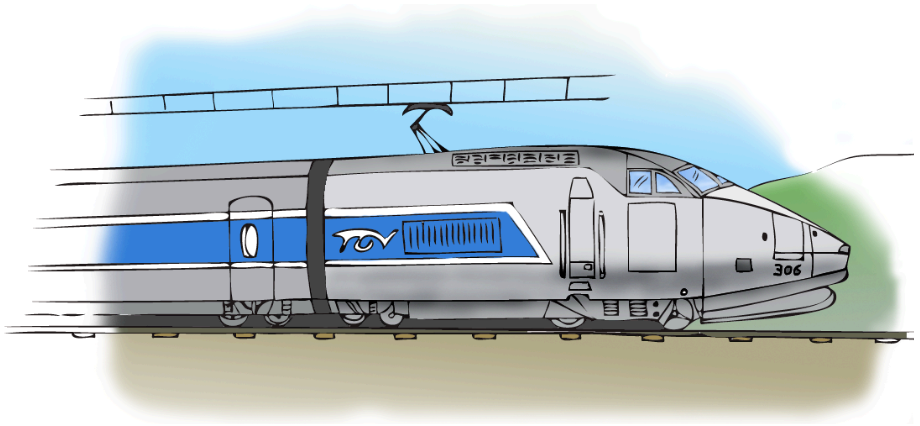
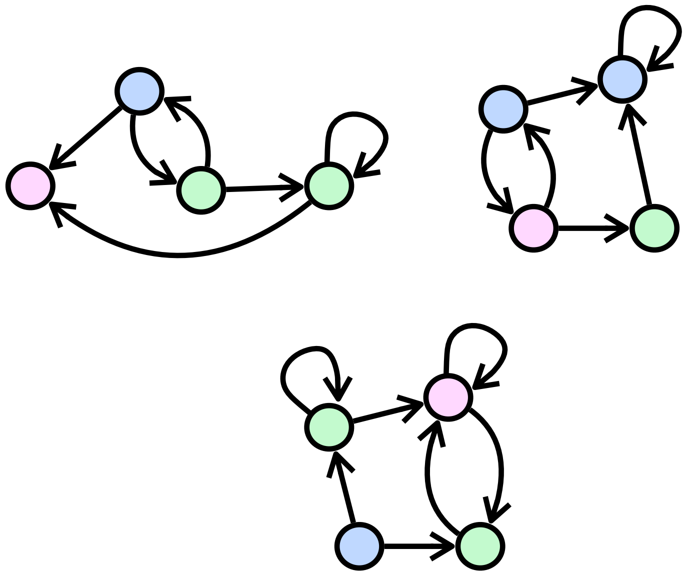
System





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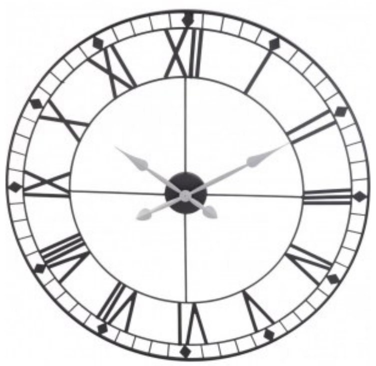

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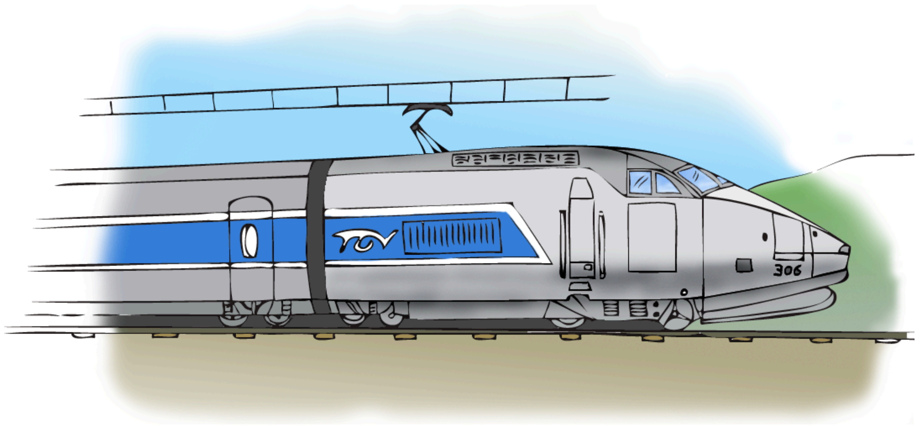
Model-checking algorithm

Yes/No/Why?

Control or synthesis

System



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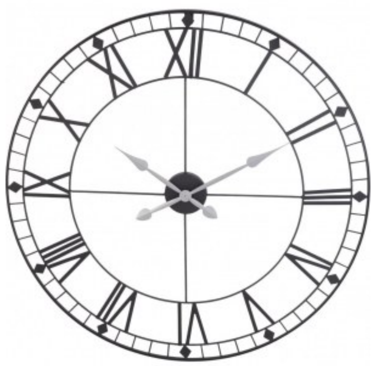

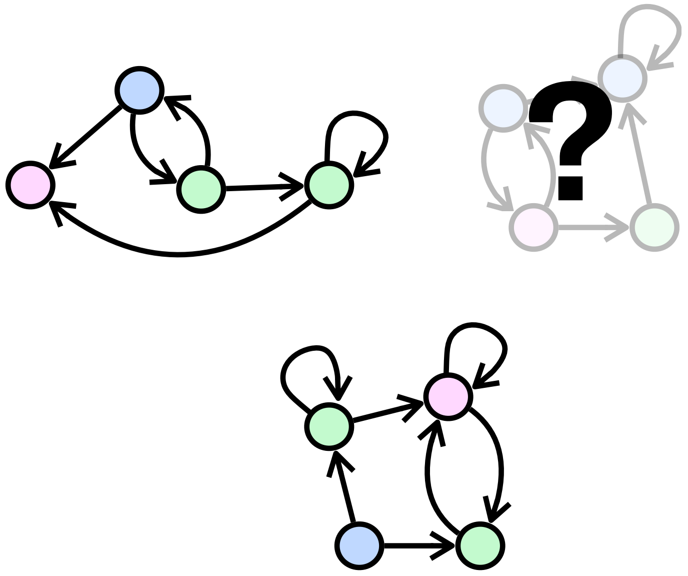

Control/synthesis algorithm

No/Yes/How?

$$\varphi = \mathbf{AG}\,\neg\text{crash} \wedge \left(\mathbb{P}(\mathbf{F}_{\leq 2h}\text{arr}) \geq 0{,}9\right)$$

The talk in one slide

Strategy synthesis for two-player games

Find good and simple controllers for systems interacting with an antagonistic environment

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Good?

Performance w.r.t. objectives / payoffs / preference relations

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Minimal information for deciding the next steps

The talk in one slide

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Minimal information for deciding the next steps

When are simple strategies sufficient to play optimally?

Our general approach

[Tho95] On the synthesis of strategies in infinite games (STACS'95).
[Tho02] Thomas. Infinite games and verification (CAV'02).
[GU08] Grädel, Ummels. Solution concepts and algorithms for infinite multiplayer games (New Perspectives in Games and Interactions, 2008).
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Our general approach

- Use **graph-based game models** (state machines) to represent the system and its evolution
- Use **game theory concepts** to express admissible situations
  - Winning strategies
  - (Pareto-)Optimal strategies
  - Nash equilibria
  - Subgame-perfect equilibria
  - ...

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Games
What they often are

Games
A broader sense

Interaction

Goal

- Model and analyze (using math. tools) situations of interactive decision making

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Ingredients

- Several decision makers (players)
- Possibly each with different goals
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Wide range of applicability

« [...] it is a context-free mathematical toolbox. »

- Social science: e.g. social choice theory
- Theoretical economics: e.g. models of markets, auctions
- Political science: e.g. fair division
- Biology: e.g. evolutionary biology
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[MSZ13] Maschler, Solan, Zamir. Game theory (2013).

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+ Computer science

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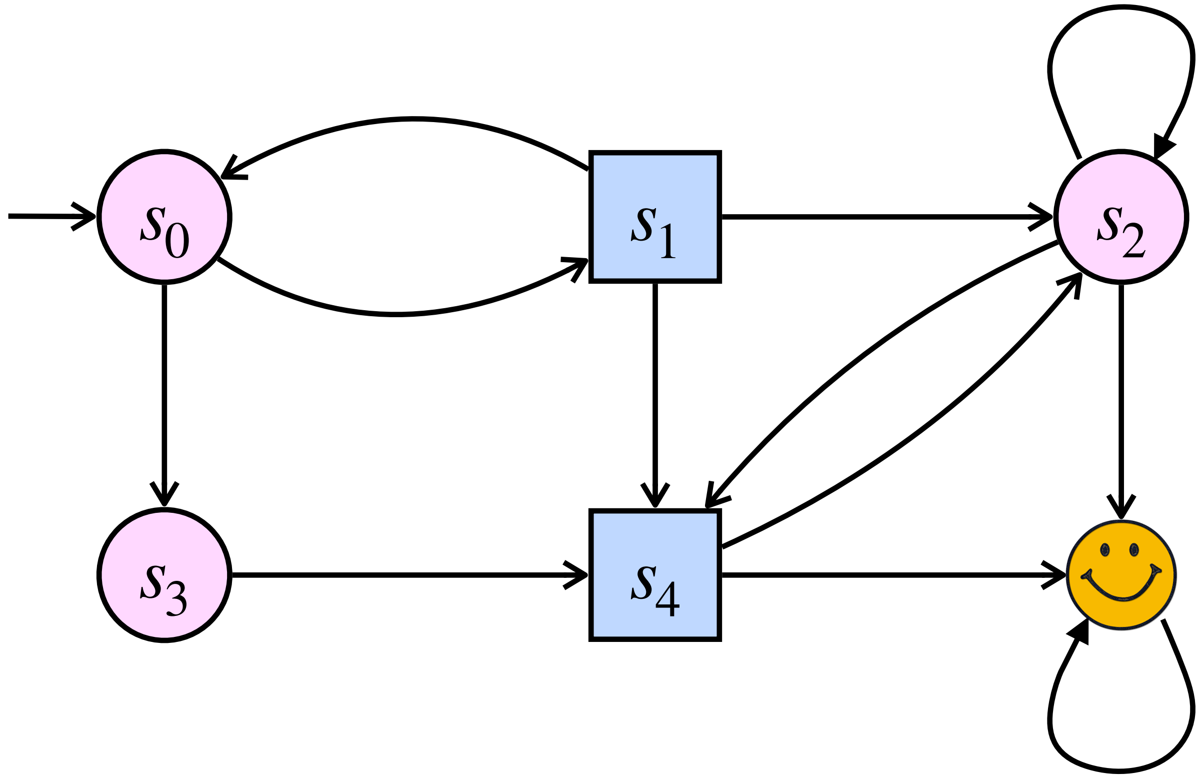
Turn-based games on graphs

$$\mathscr{G} = (\overset{\text{States}}{S}, s_0, S_1, S_2, \overset{\text{Edges}}{E})$$

○ : player P_1

□ : player P_2



Turn-based games on graphs

States Edges

$$\mathcal{G} = (S, s_0, S_1, S_2, E)$$

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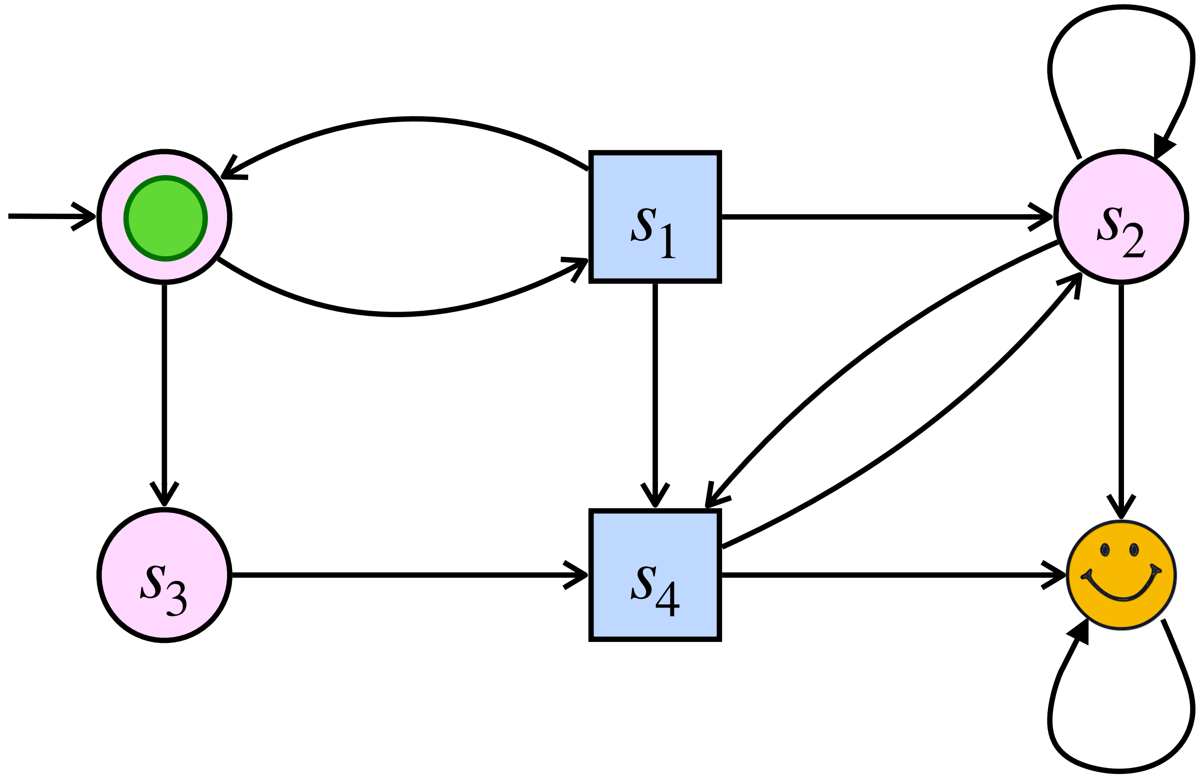

s_0

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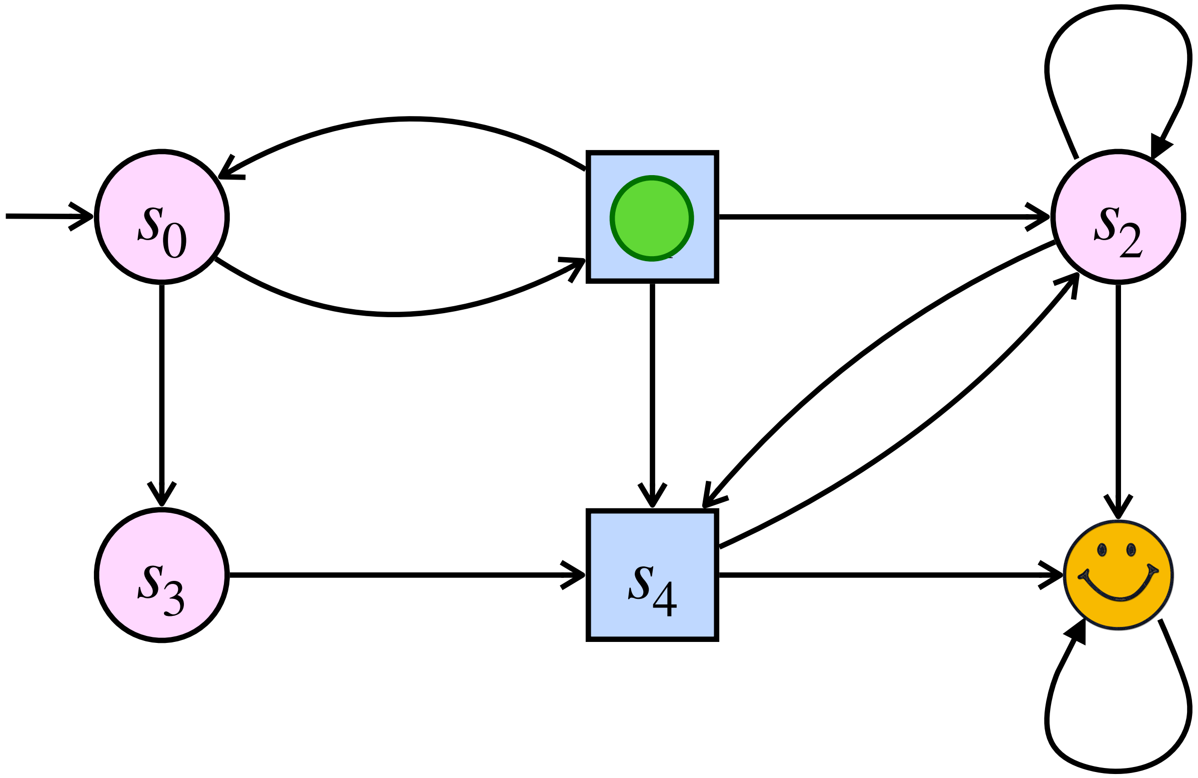

$$s_0 \rightarrow s_1$$

1. P_1 chooses the edge (s_0, s_1)

Turn-based games on graphs

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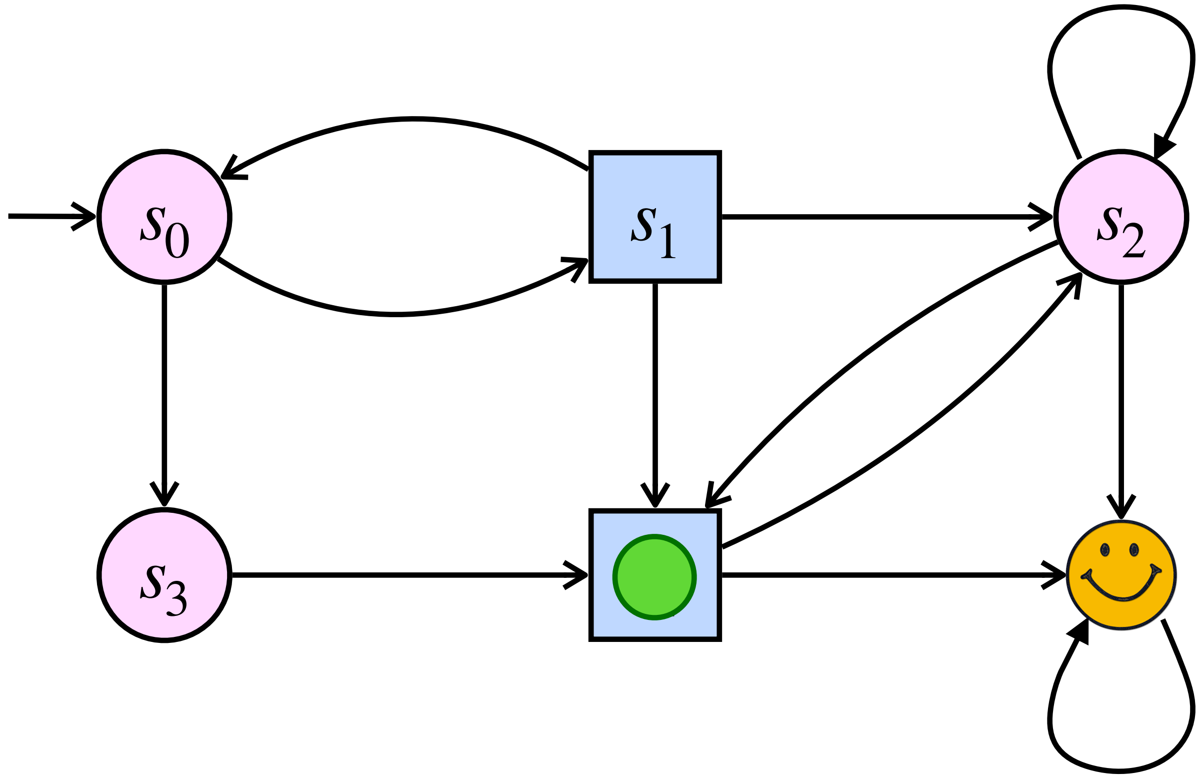

$$s_0 \rightarrow s_1 \rightarrow s_4$$

1. P_1 chooses the edge (s_0, s_1)
2. P_2 chooses the edge (s_1, s_4)

Turn-based games on graphs

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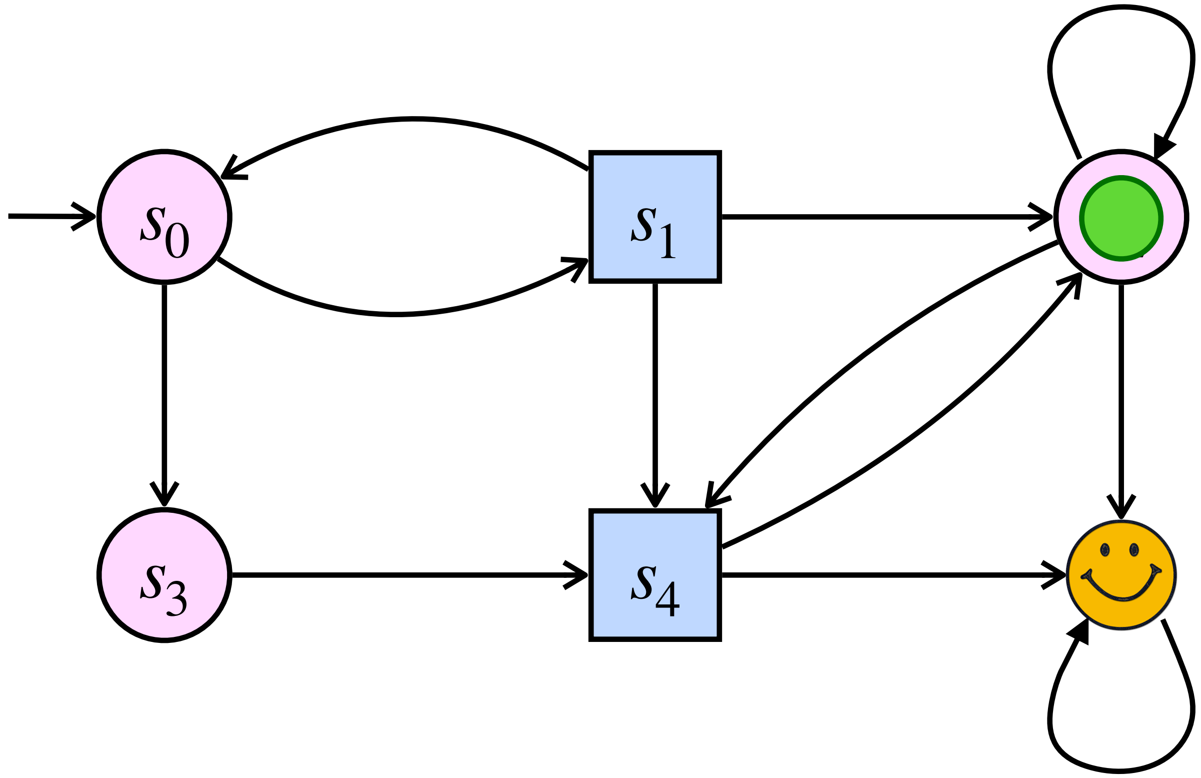

$$s_0 \to s_1 \to s_4 \to s_2$$

1. P_1 chooses the edge (s_0, s_1)
2. P_2 chooses the edge (s_1, s_4)
3. P_2 chooses the edge (s_4, s_2)

Turn-based games on graphs

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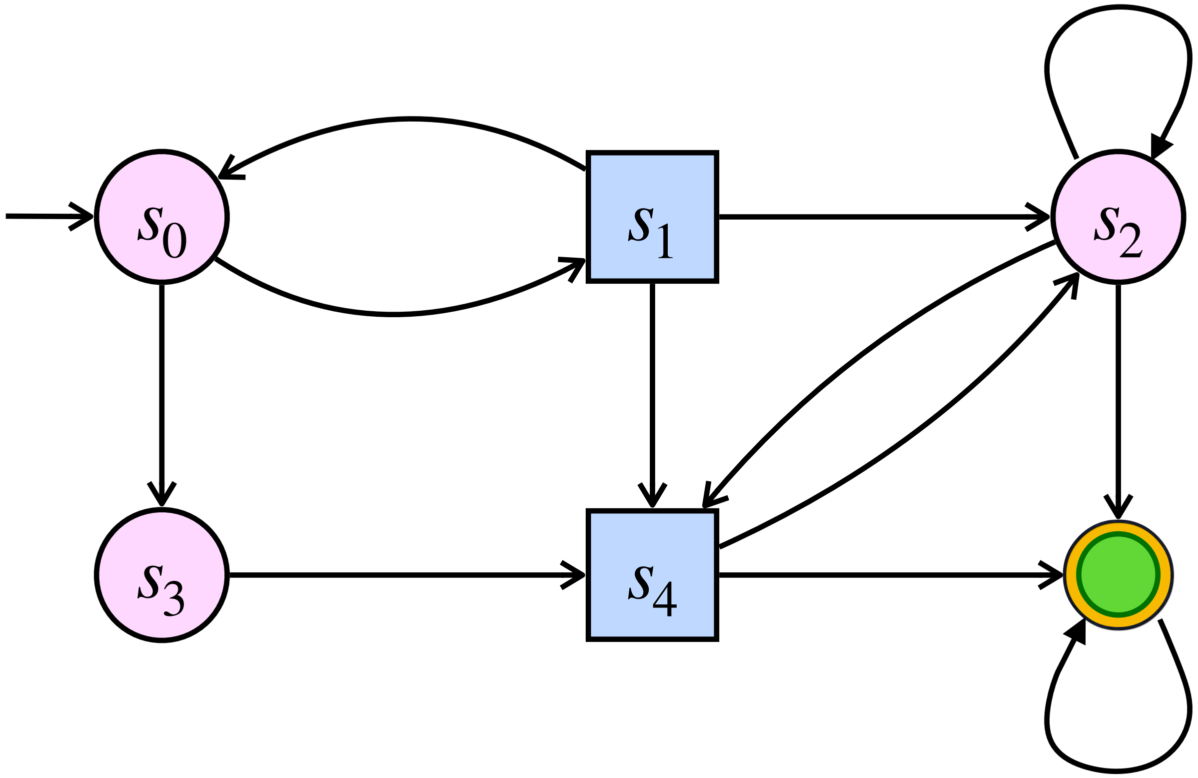

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4. P_1 chooses the edge $(s_2,$ ☺$)$

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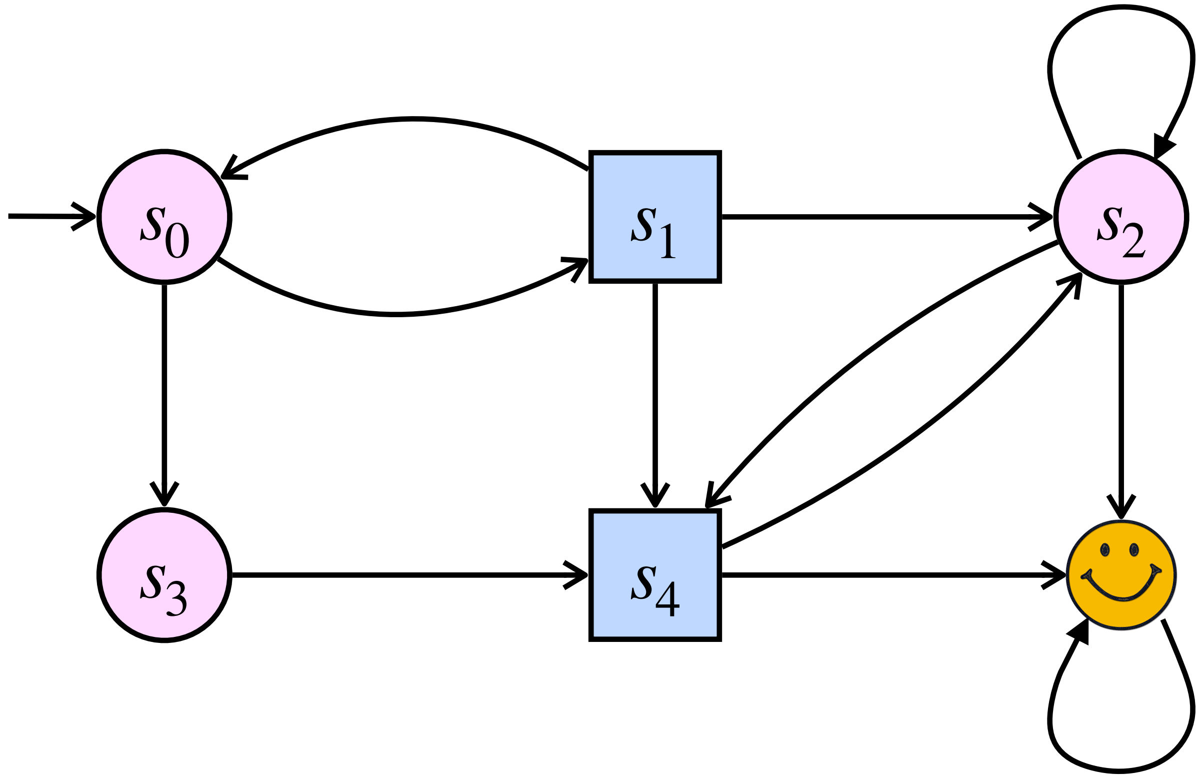

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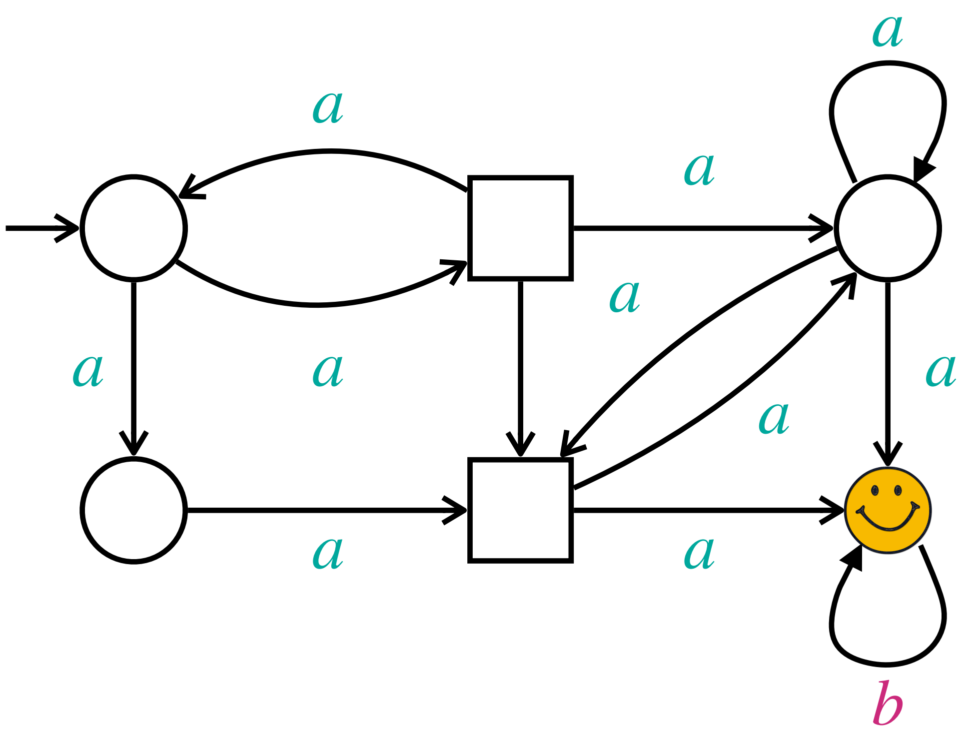
Players use **strategies** to play.
A strategy for P_i is $\sigma_i : S^*S_i \to E$

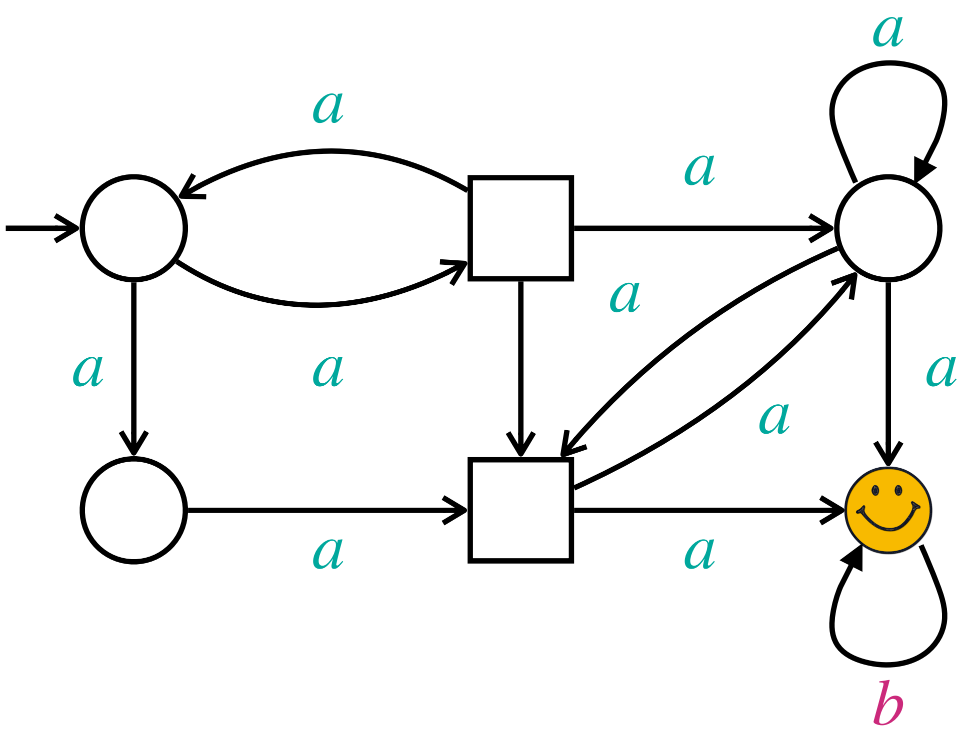
Objectives for the players



$C = \{a, b\}$ set of colors

$E \subseteq S \times C \times S$

Objectives for the players



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- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

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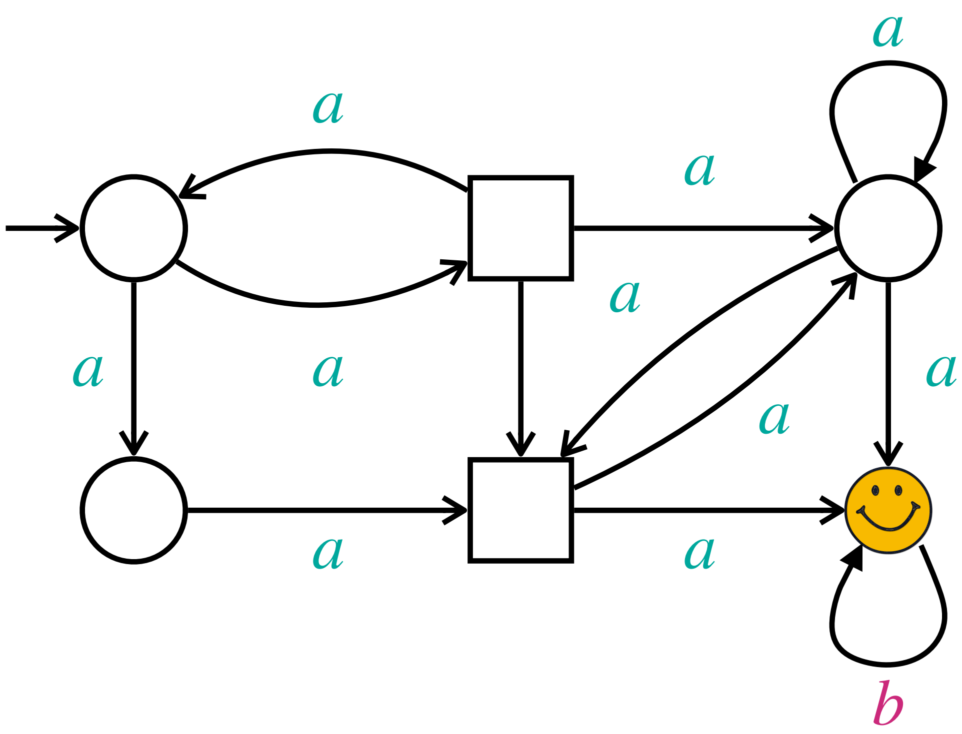

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- Payoff function: $p_i \colon C^\omega \to \mathbb{R}$, e.g. mean-payoff

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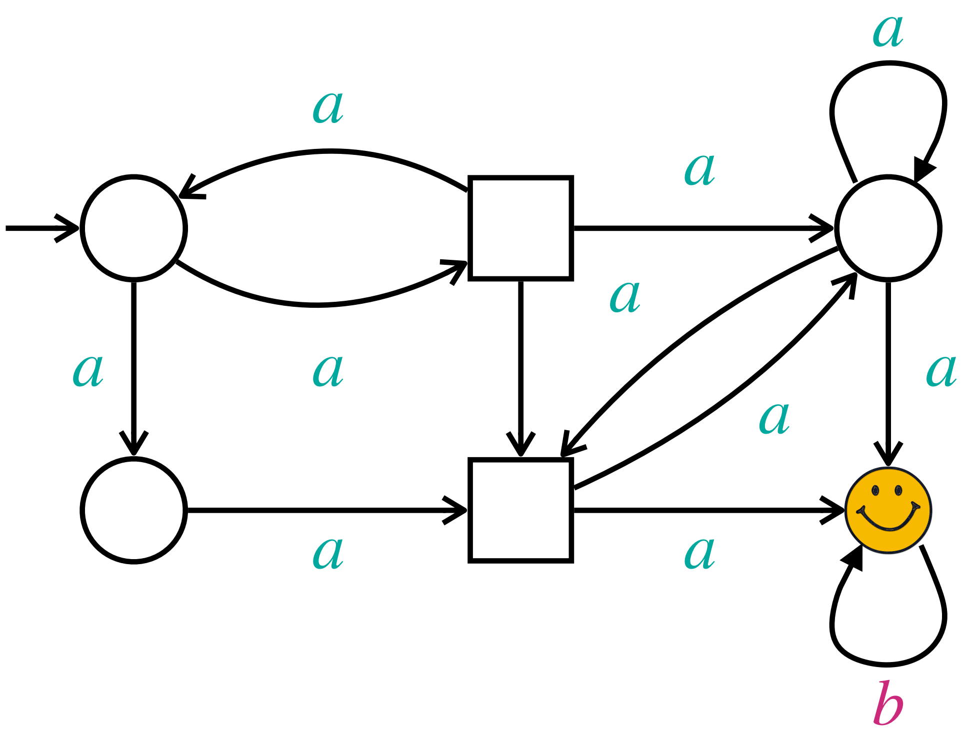

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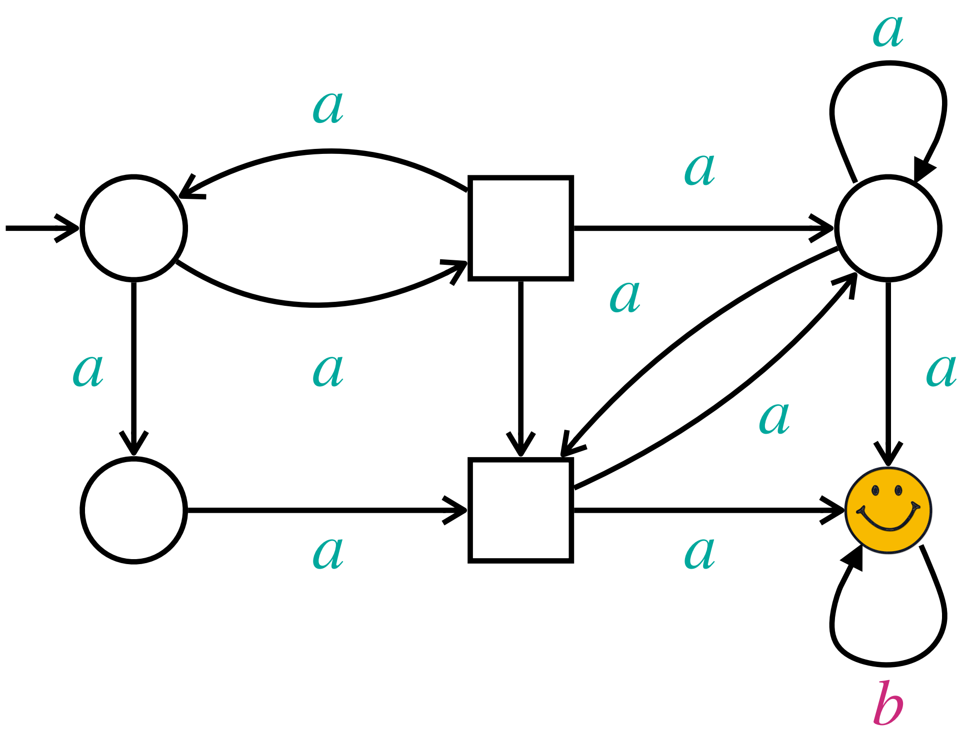
- Payoff function: $p_i : C^\omega \to \mathbb{R}$, e.g. mean-payoff
- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$ (total preorder)

Objectives for the players



Zero-sum assumption

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- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

$W_2 = W_1^c$

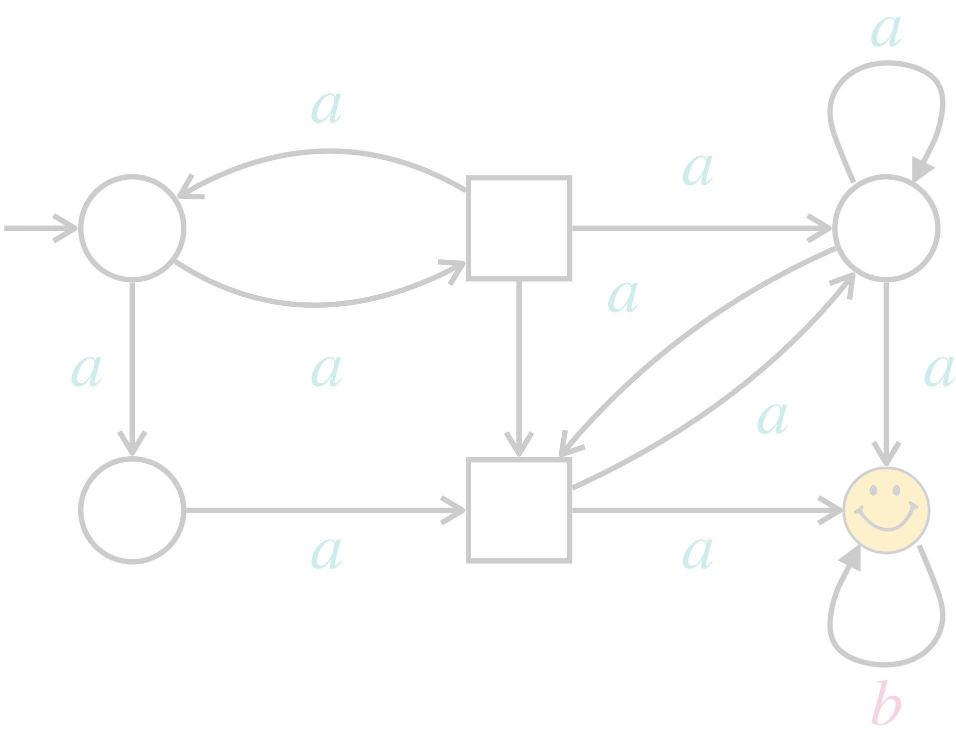
- Payoff function: $p_i : C^\omega \to \mathbb{R}$, e.g. mean-payoff

$p_1 + p_2 = 0$

- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$
(total preorder)

$\sqsubseteq_2 = \sqsubseteq_1^{-1}$

Objectives for the players



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- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

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- We focus on winning objectives, and write W for W_1

0

- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$ (total preorder)

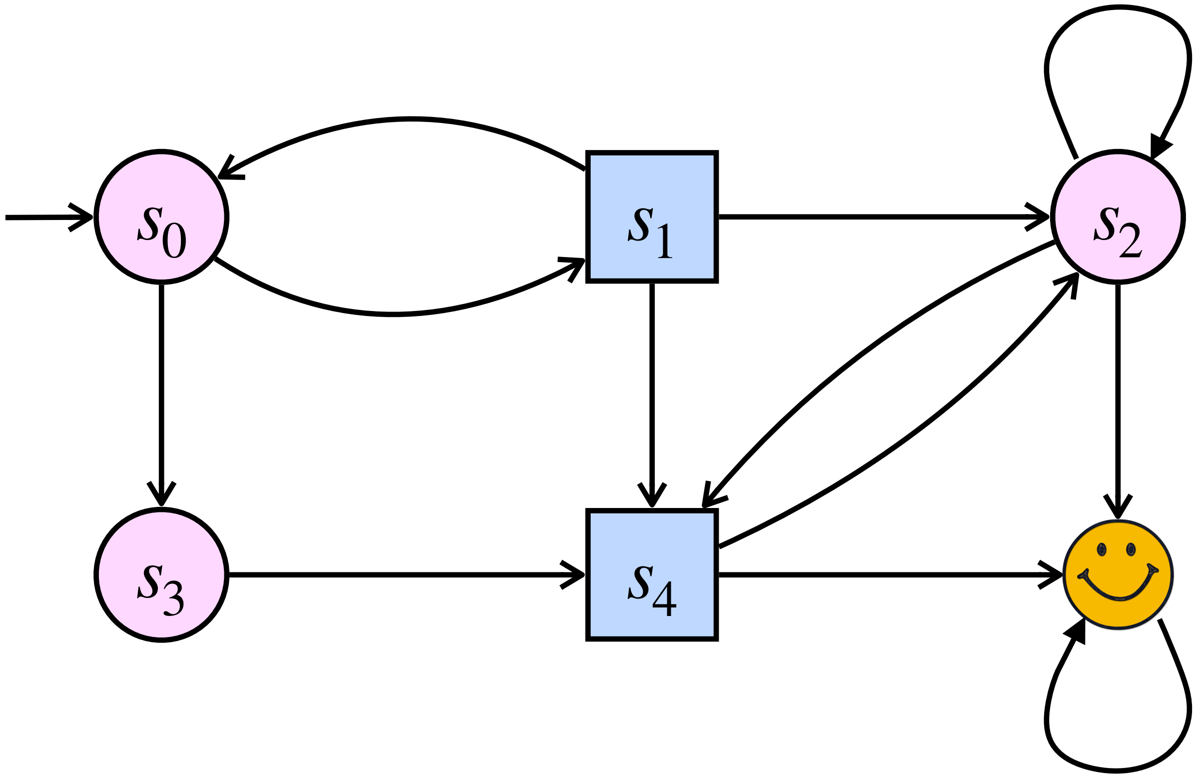
$\sqsubseteq_2 = \sqsubseteq_1^{-1}$

What does it mean to win a game?

What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\left(s_0 s_1 \ldots s_j\right)$. We write Out$(\sigma_i)$.

Outcomes of a strategy



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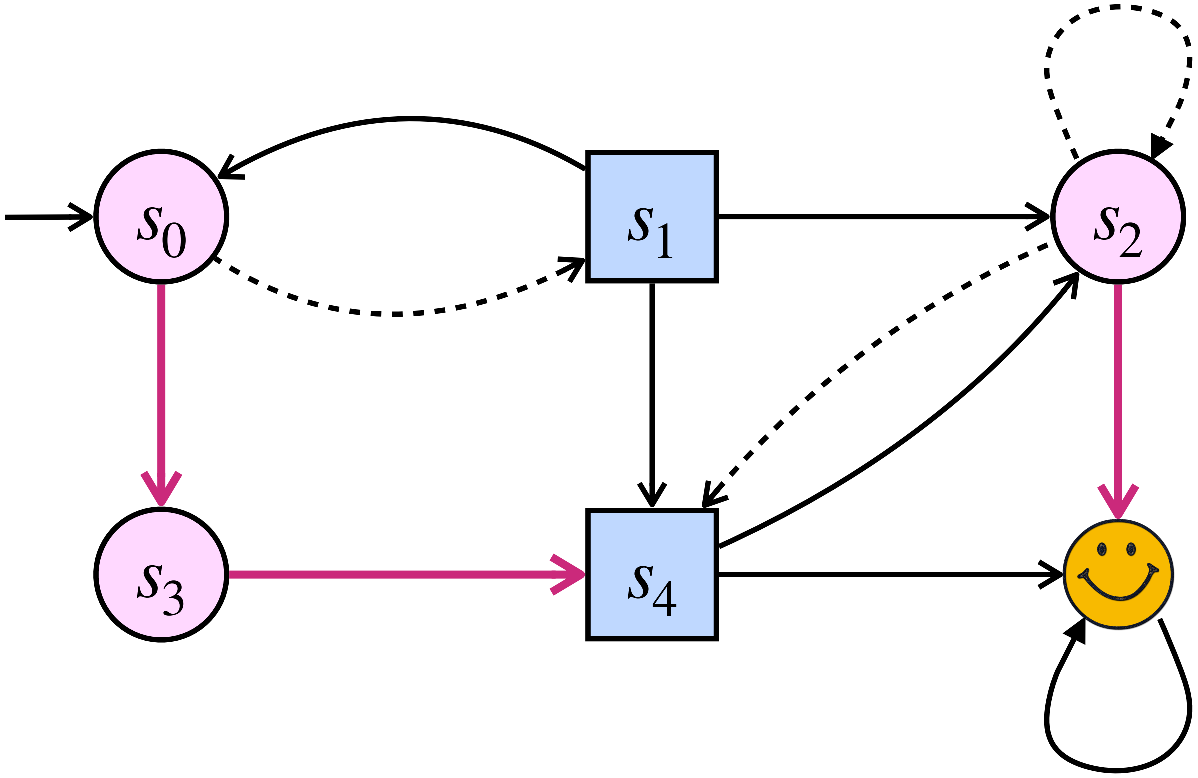
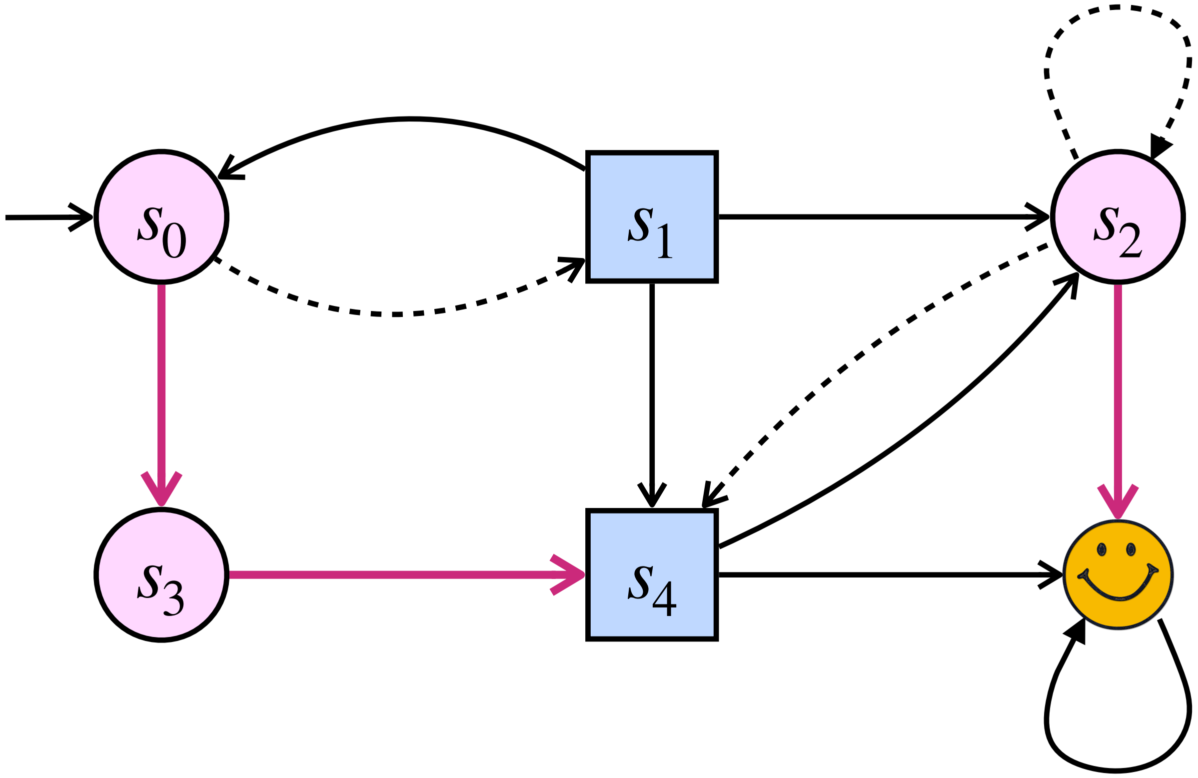
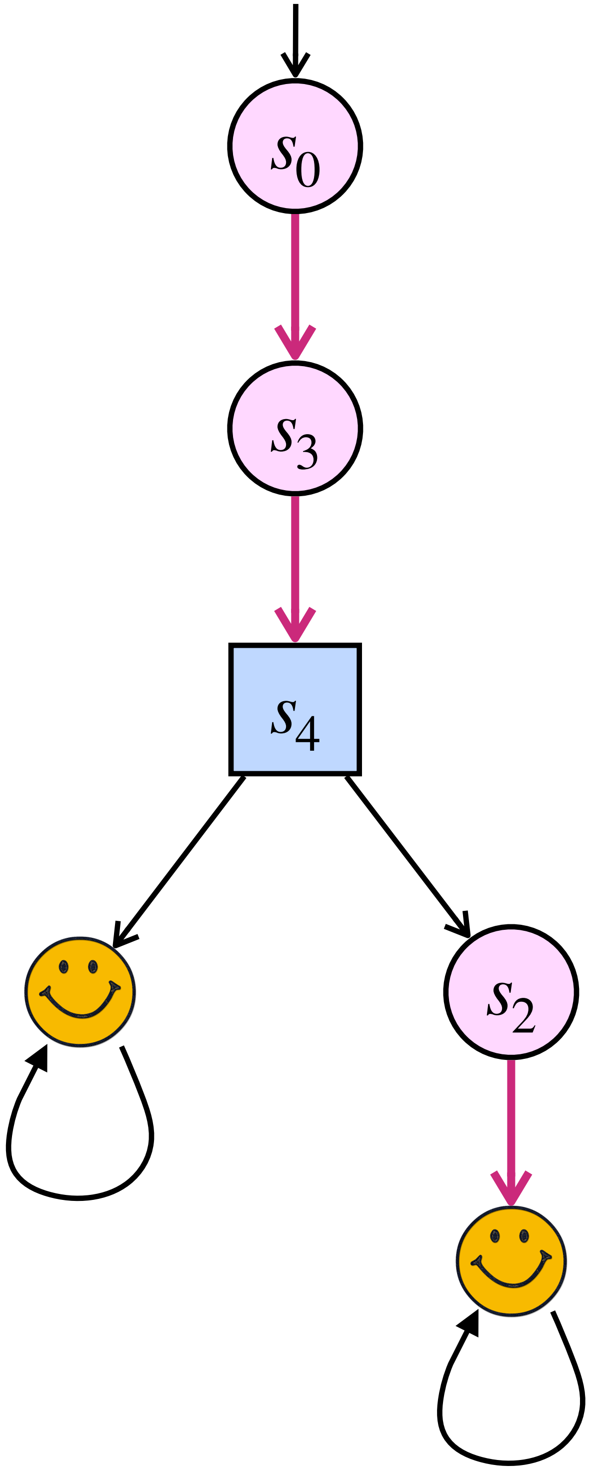


- Strategy σ

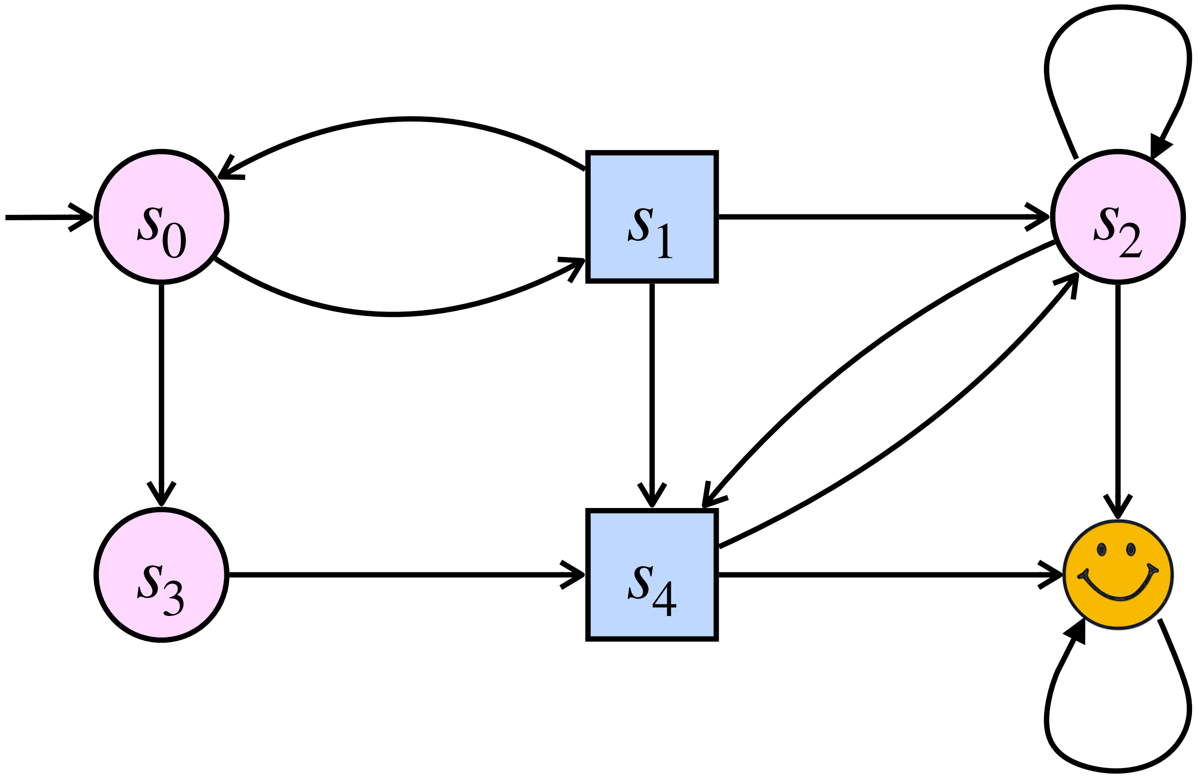
Outcomes of a strategy





- Strategy σ
- Out(σ) has two plays, which are both winning

Outcomes of a strategy



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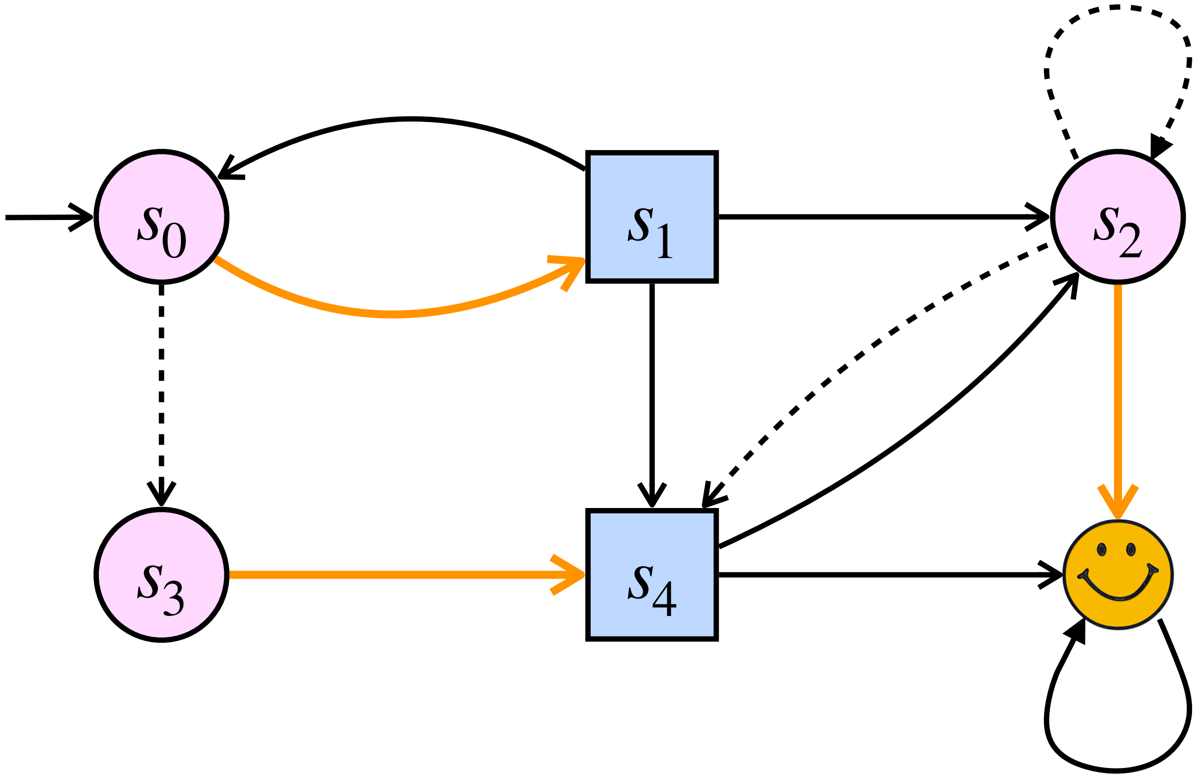

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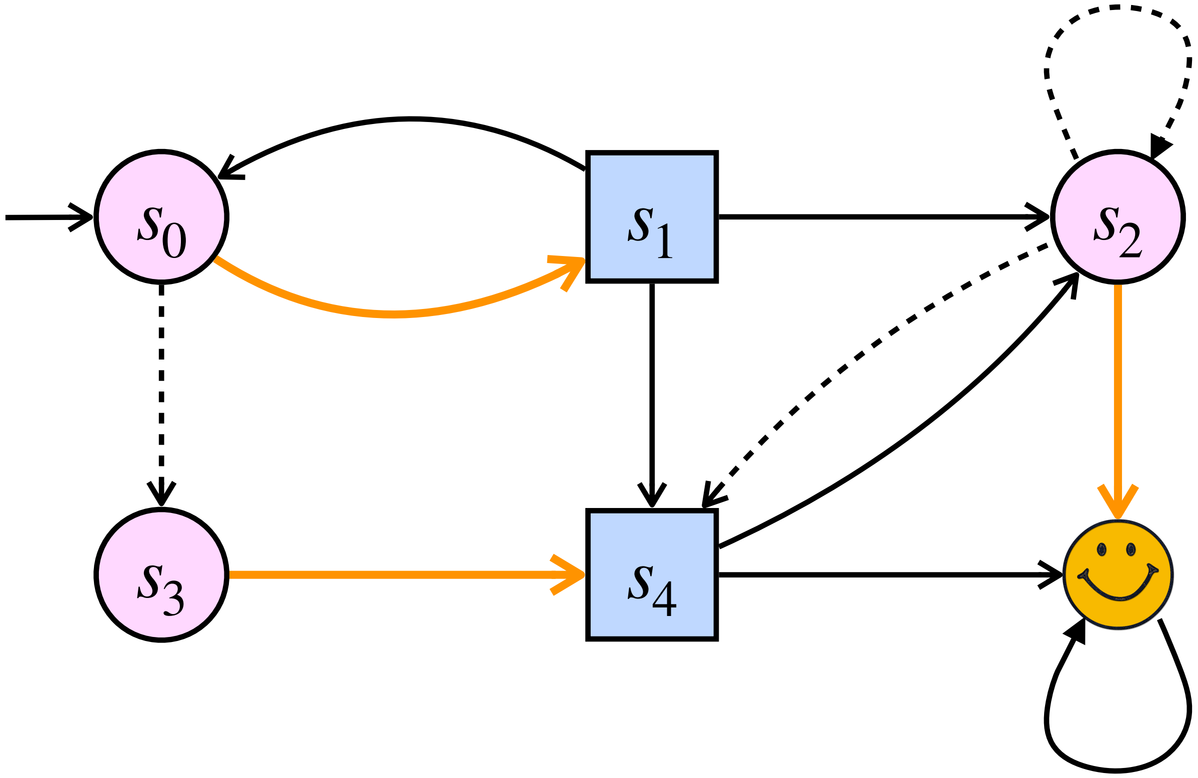

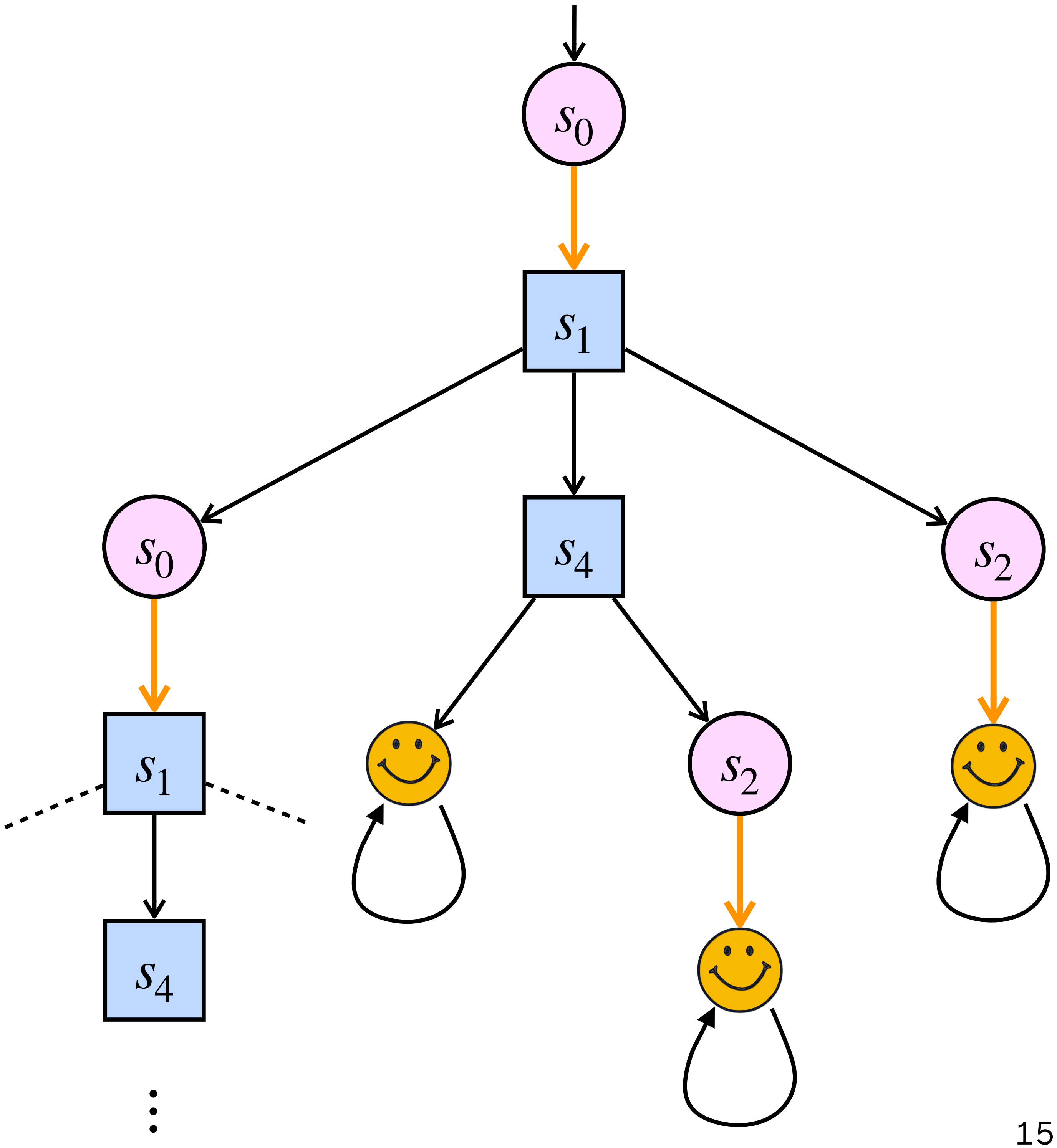

- Strategy σ
- Out(σ) has infinitely many plays, some of them are not winning

What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\left(s_0 s_1 \ldots s_j\right)$. We write Out$(\sigma_i)$.

- σ_i is **winning** if all plays compatible with σ_i belong to W_i
  σ_i is **optimal** if it is winning or if the initial state is losing

[Mar75] Martin. Borel determinacy (Annals of Mathematics).

What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\big(s_0 s_1 \ldots s_j\big)$. We write Out$(\sigma_i)$.

- σ_i is **winning** if all plays compatible with σ_i belong to W_i
  σ_i is **optimal** if it is winning or if the initial state is losing

Martin's determinacy theorem

Turn-based zero-sum games are determined for Borel winning objectives: in every game, either P_1 or P_2 has a winning strategy.

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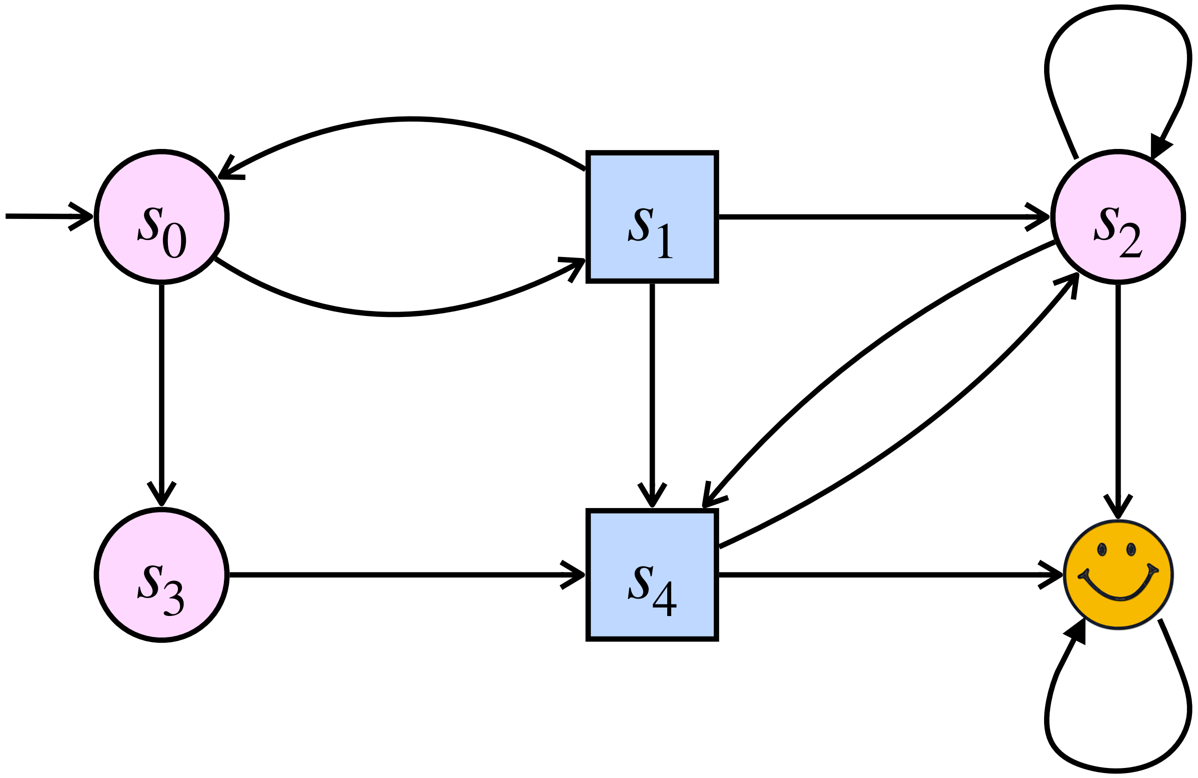
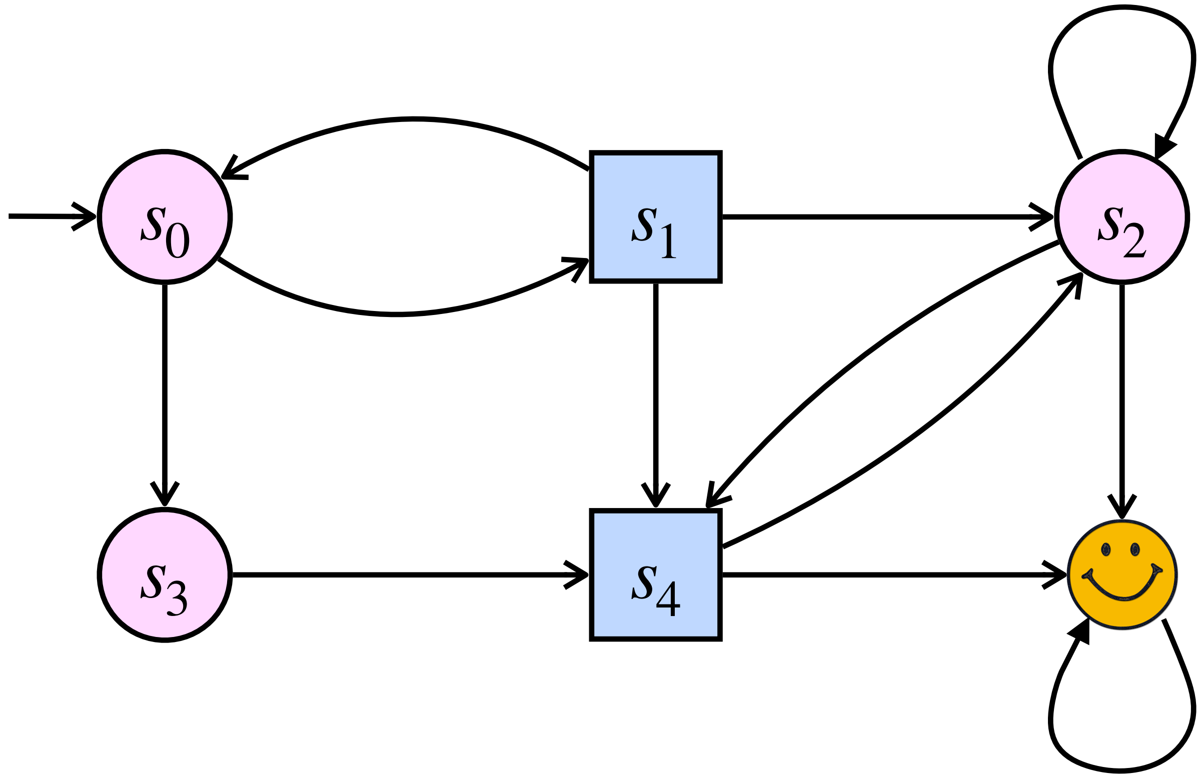
Relevant questions



φ = Reach(☺)

Relevant questions



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- Can P_1 win the game, i.e. does P_1 have a winning strategy?

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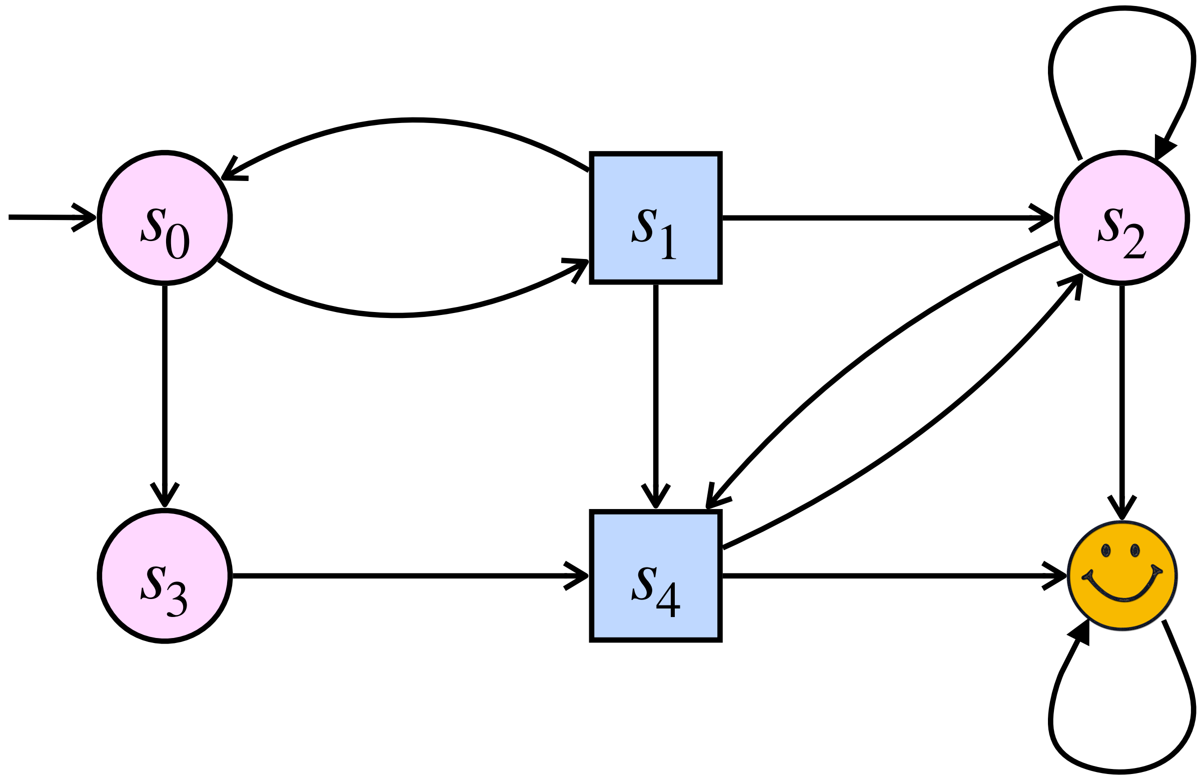

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- Can P_1 win the game, i.e. does P_1 have a winning strategy?
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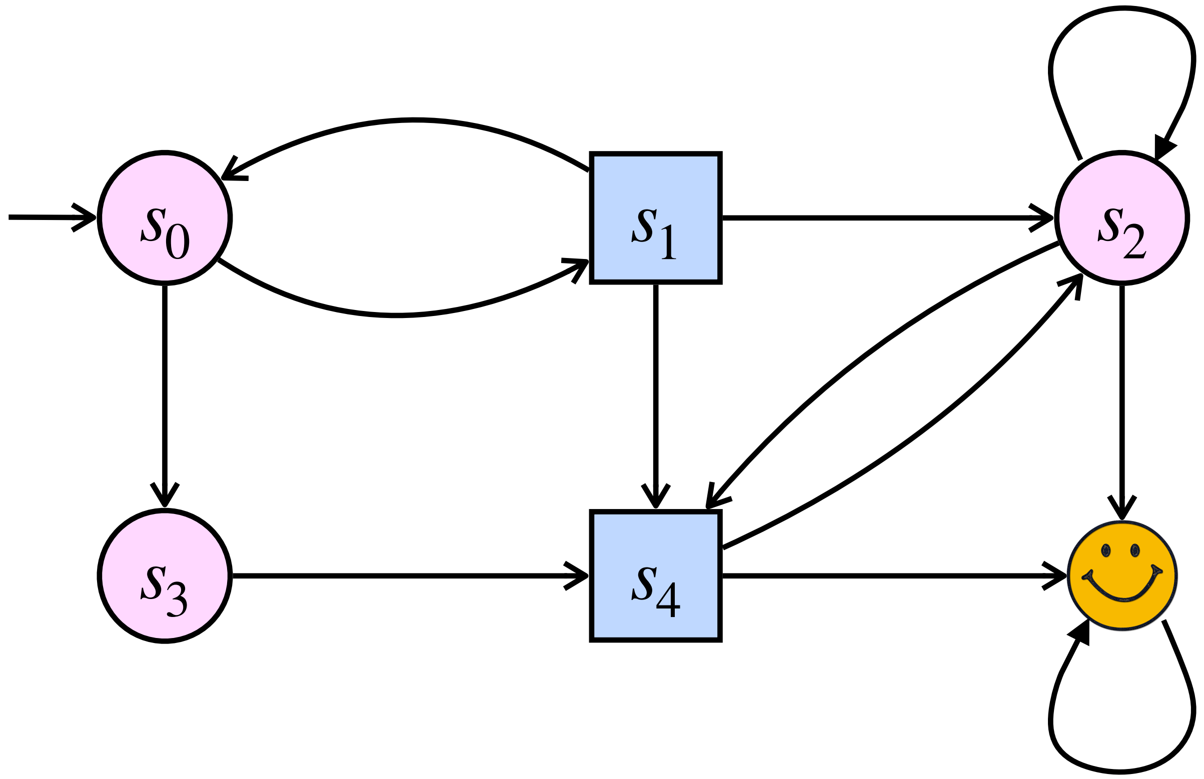

φ = Reach(☺)

- Can P_1 win the game, i.e. does P_1 have a winning strategy?
- Is there an effective (efficient) way of winning?
- How complex is it to win?

Example: the Nim game

- Players alternate
- Each player can take one or two sticks
- The player who takes the last one wins
- P_1 starts

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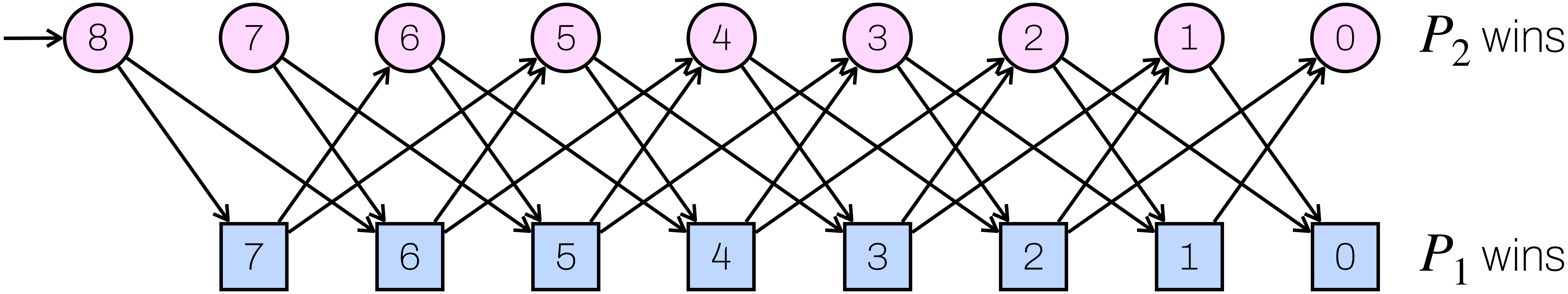

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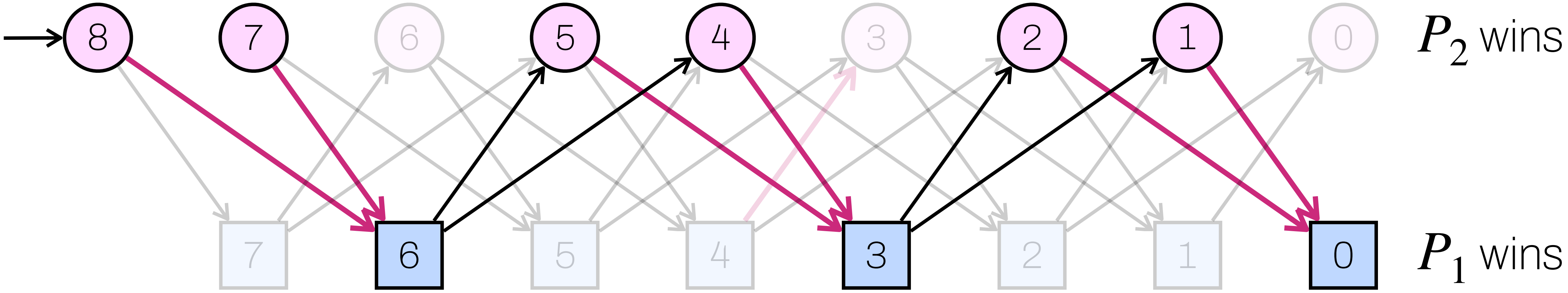

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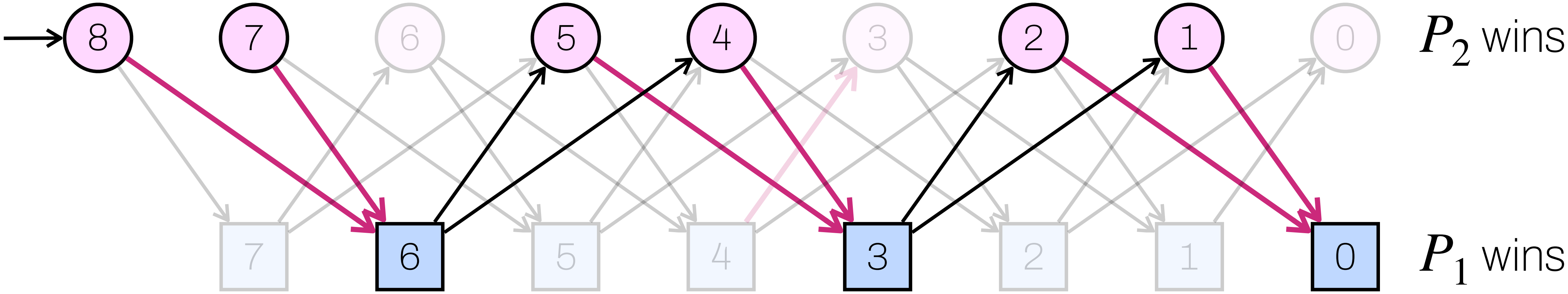

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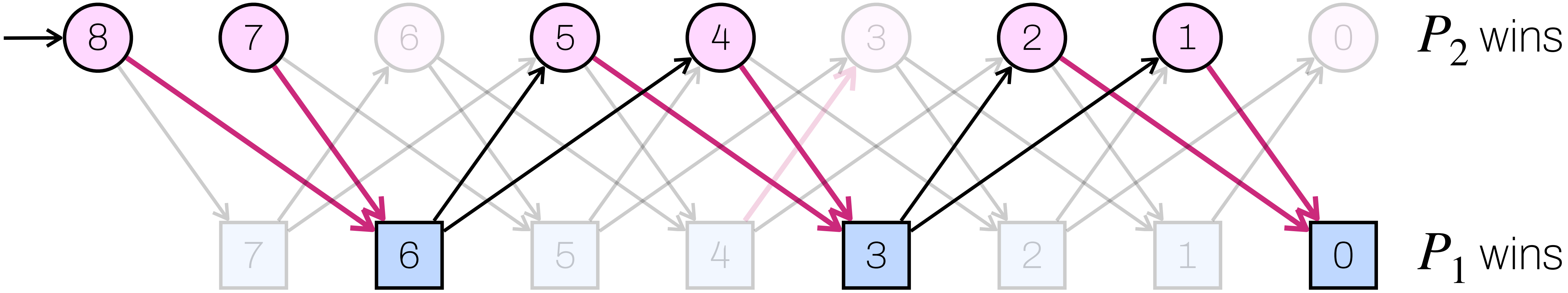


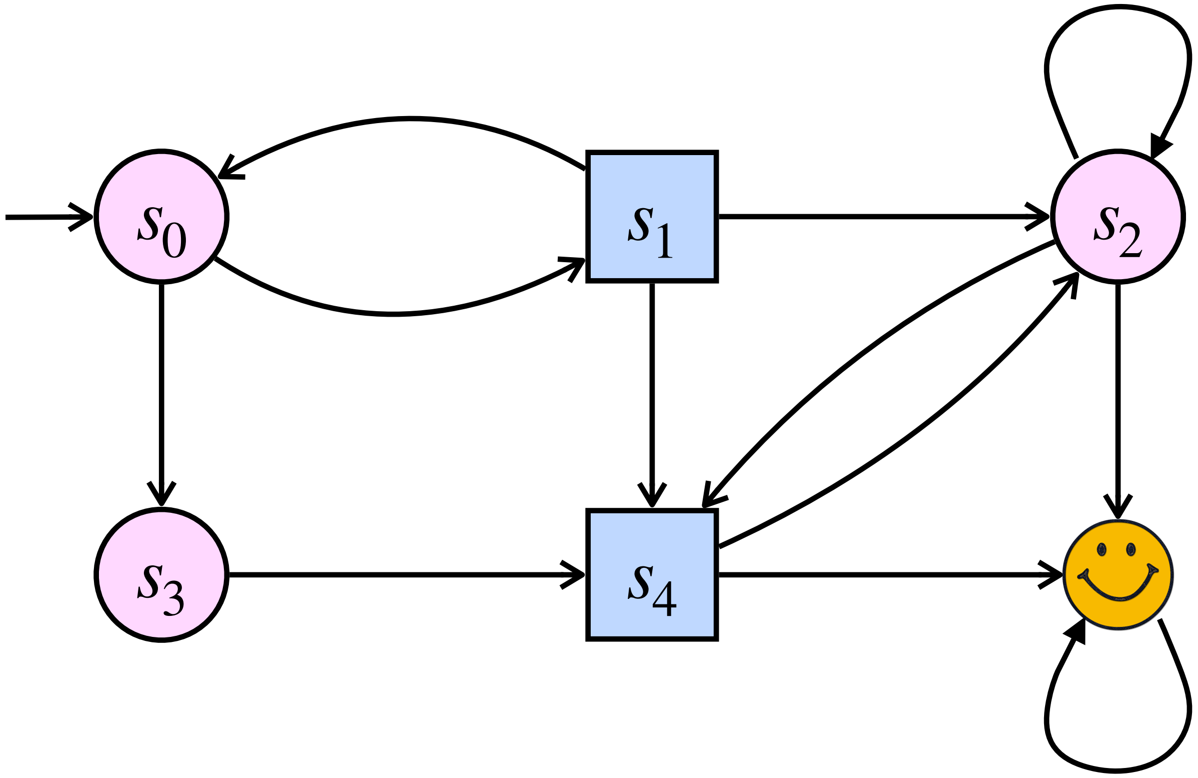
P_1 wins

- from all ○ $\equiv 1$ or $2 \mod 3$
- from all □ $\equiv 0 \mod 3$

P_2 wins

- from all ○ $\equiv 0 \mod 3$
- from all □ $\equiv 1$ or $2 \mod 3$

Computation of winning states in the running example



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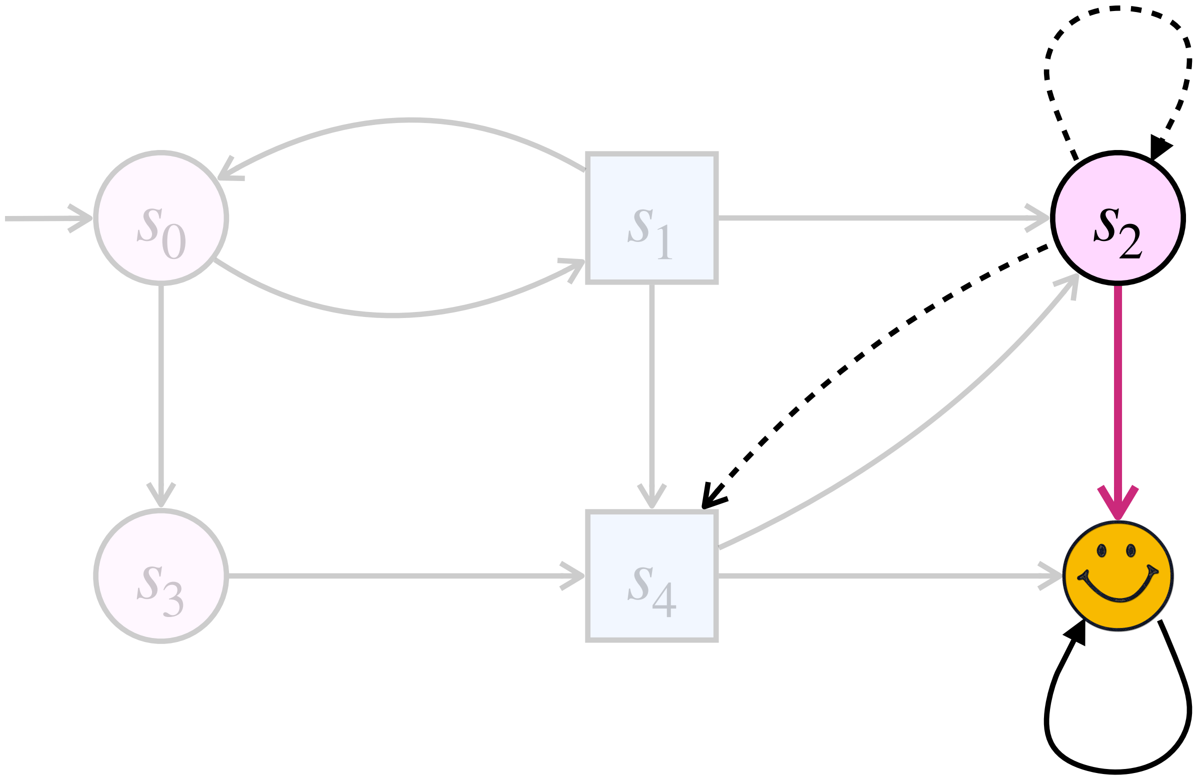

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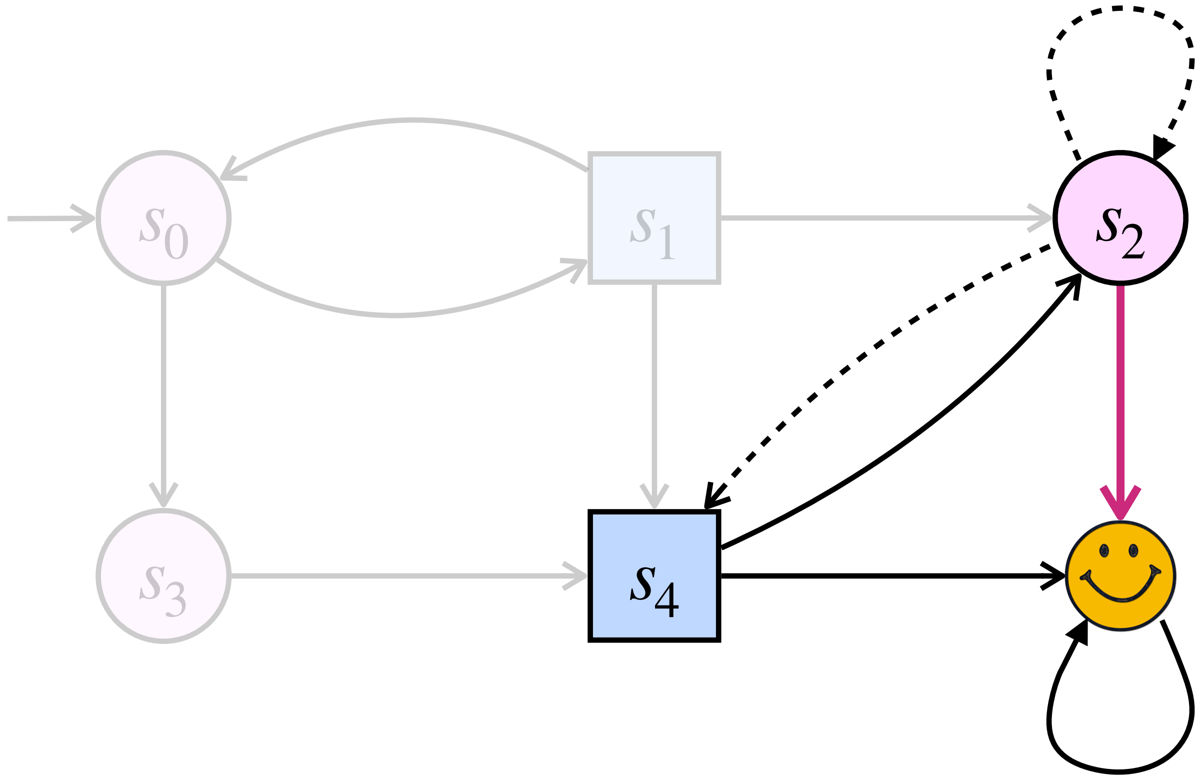

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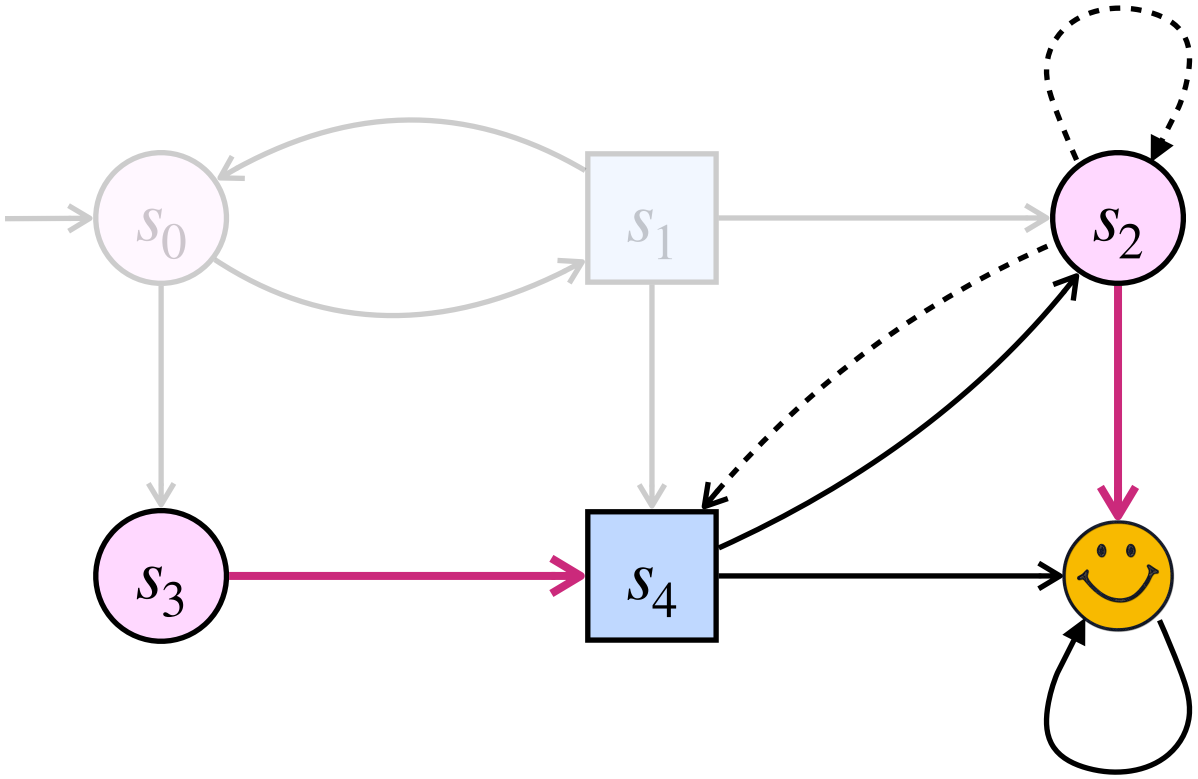

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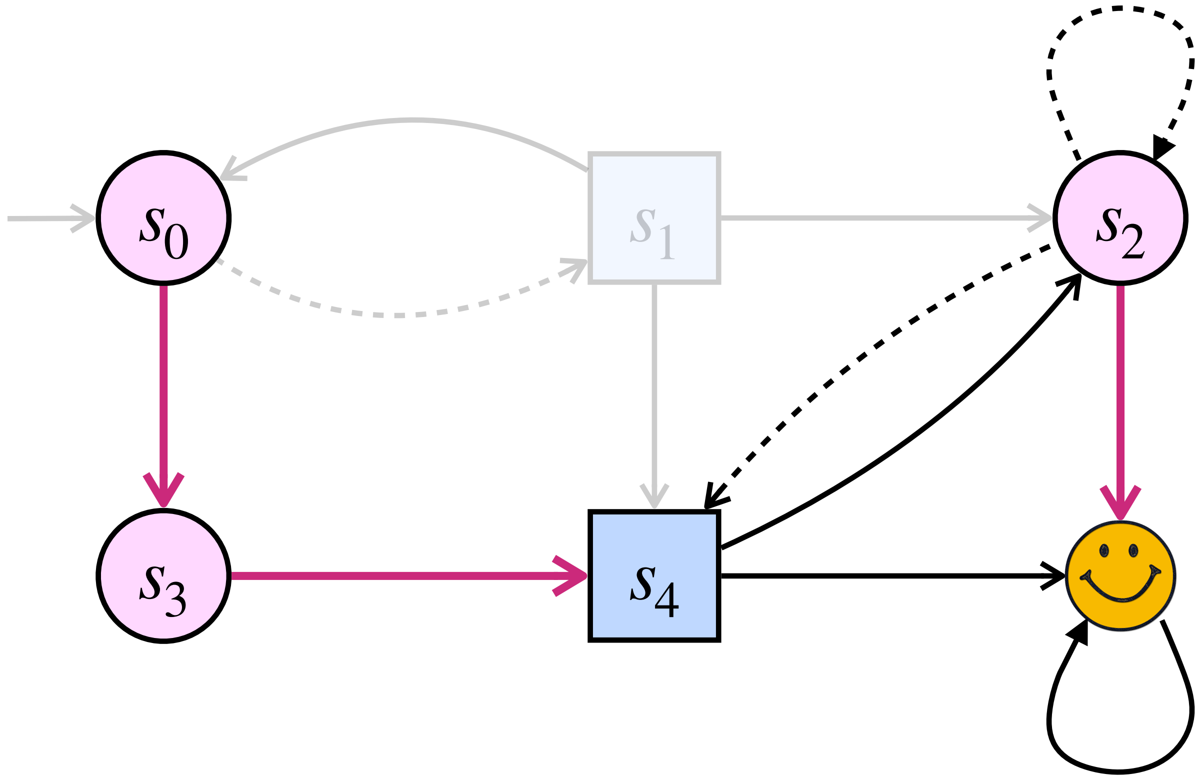

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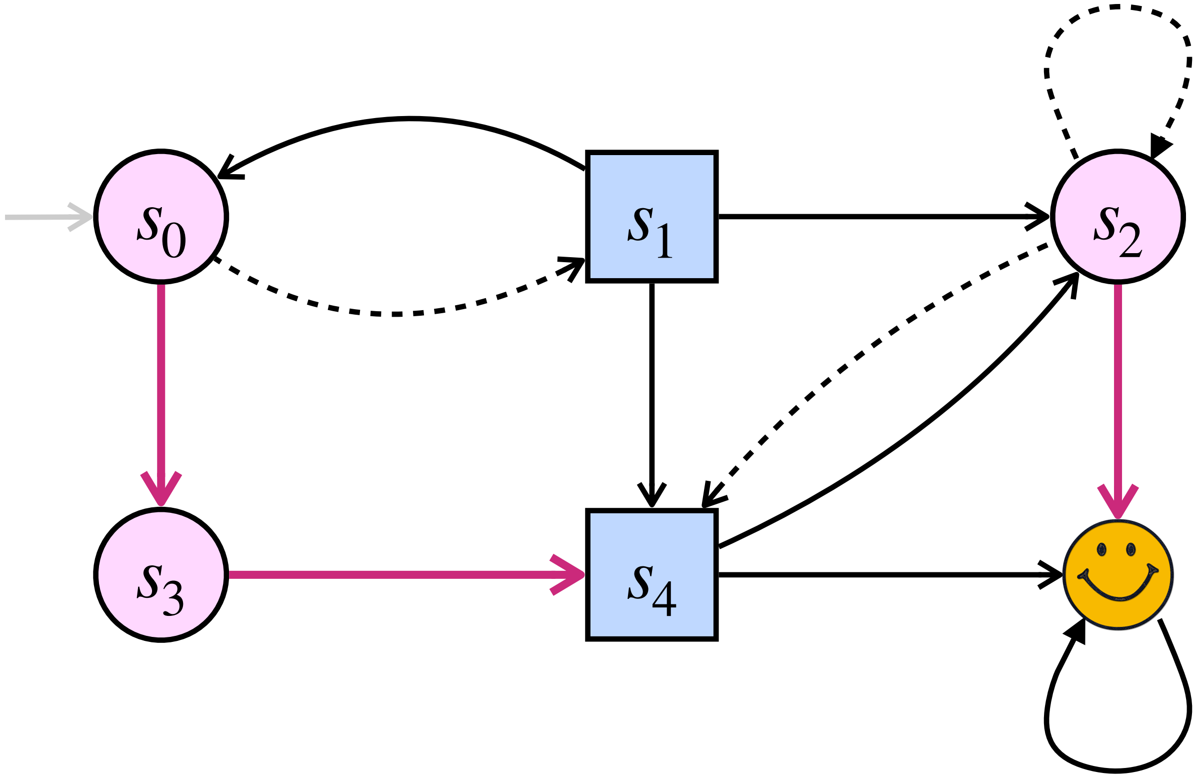

All states are winning for P_1

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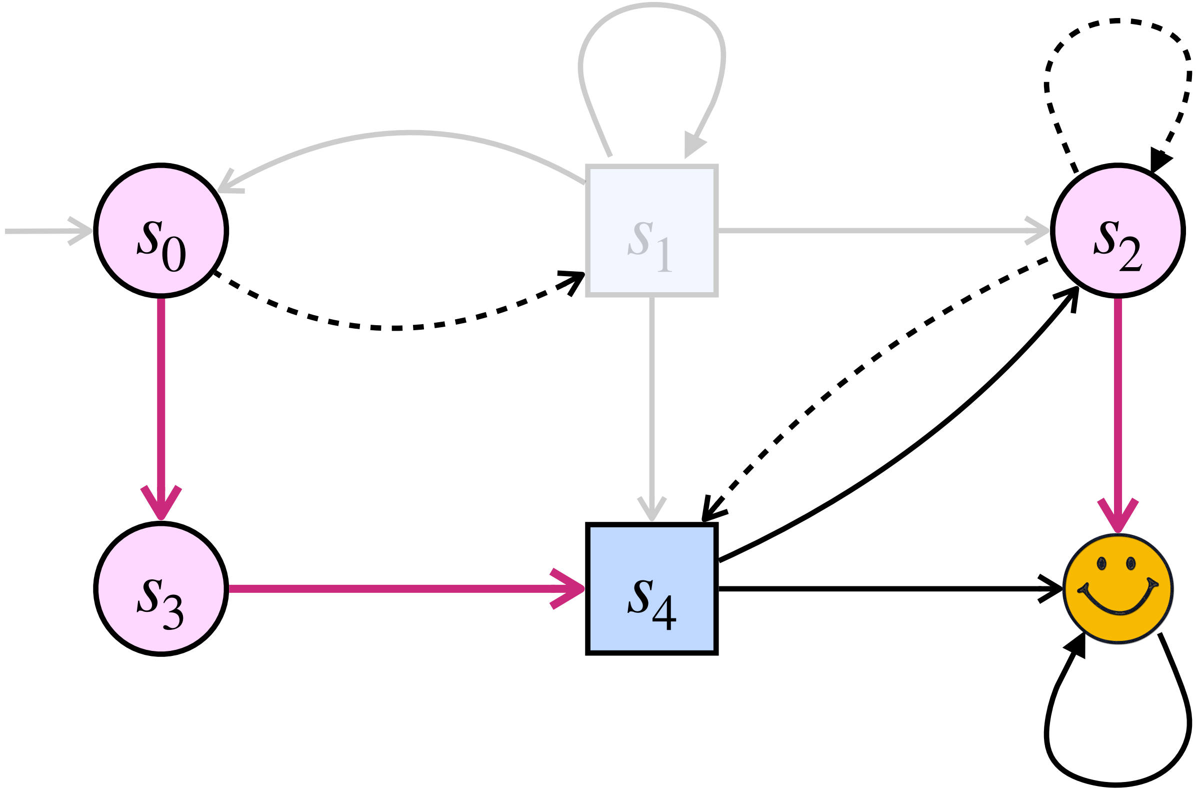

One state is not winning for P_1

It is winning for P_2

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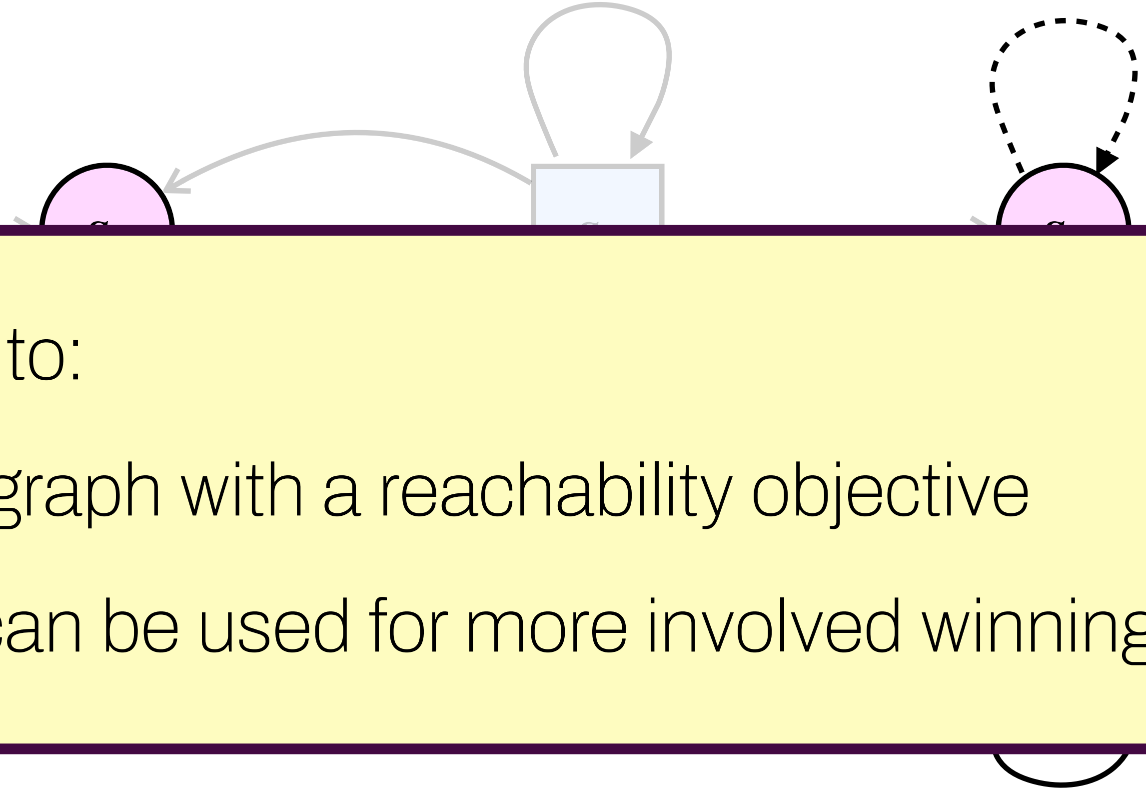

- This generalizes to:
  - Any game on graph with a reachability objective
  - Similar ideas can be used for more involved winning objectives

One state is not winning for P_1
It is winning for P_2

Limits — Chess game

[Zer13] Zermelo. Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels (Congress Mathematicians, 1912).
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Limits — Chess game

Zermelo's Theorem

In chess either white can force a win, or black can force a win, or both can force at least a draw.

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- According to Claude Shannon, there are 10^{43} legit positions in chess

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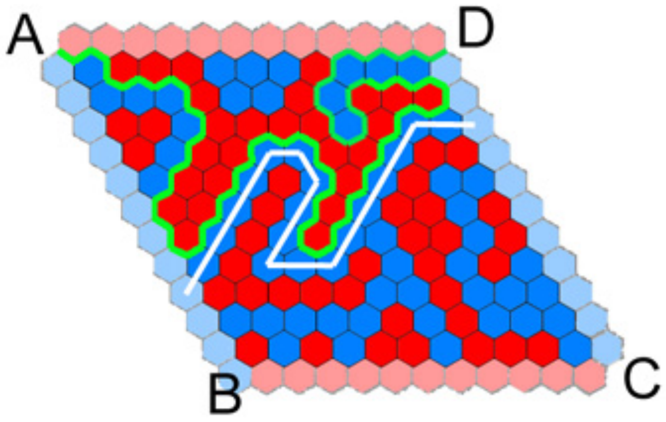

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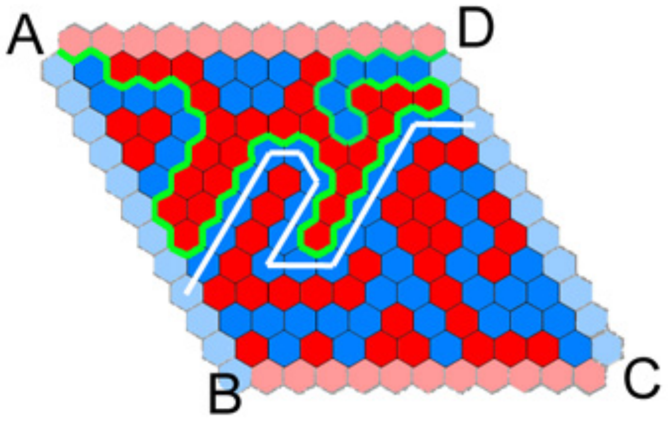

Solving the Hex game

First player has always a winning strategy.

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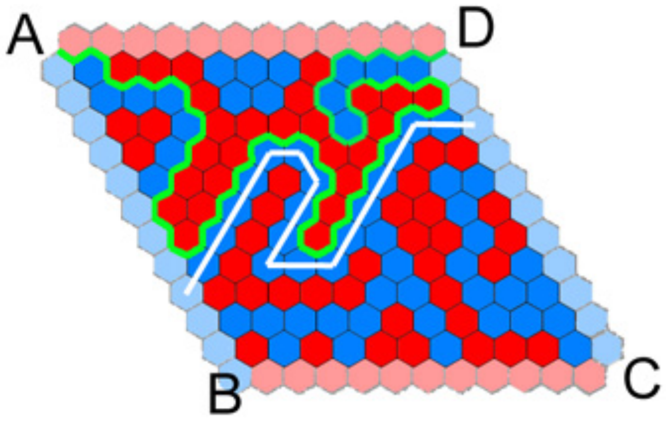

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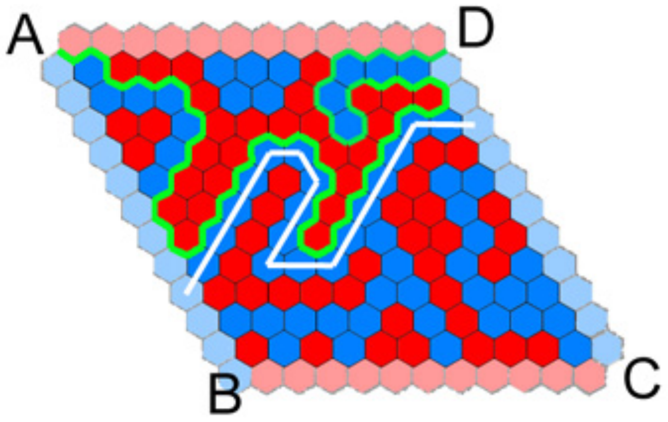


Solving the Hex game

First player has always a winning strategy.

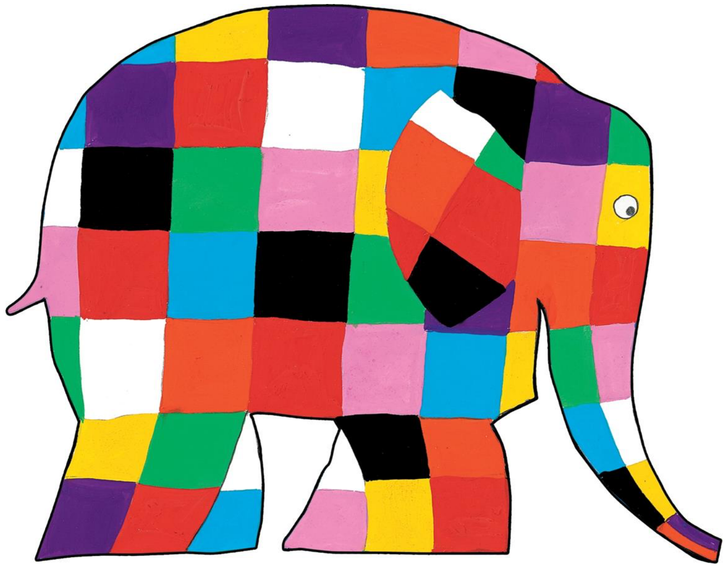
- Determinacy results (no tie is possible) + strategy stealing argument
- A winning strategy is not known yet (for boards of size ≥ 13)

What we do not consider

- Concurrent games
- Stochastic games and stochastic strategies
  - Values
  - Determinacy of Blackwell games
- Partial information

Families of strategies

Families of strategies



General strategies

$$\sigma_i : S^* S_i \to E$$

- May use any information of the past execution
- Information used is therefore potentially infinite
- Not adequate if one targets implementation

On the simplest side: positional strategies

From $\sigma_i : S^*S_i \to E$ to $\sigma_i : S_i \to E$

On the simplest side: positional strategies

From $\sigma_i : S^*S_i \rightarrow E$ to $\sigma_i : S_i \rightarrow E$

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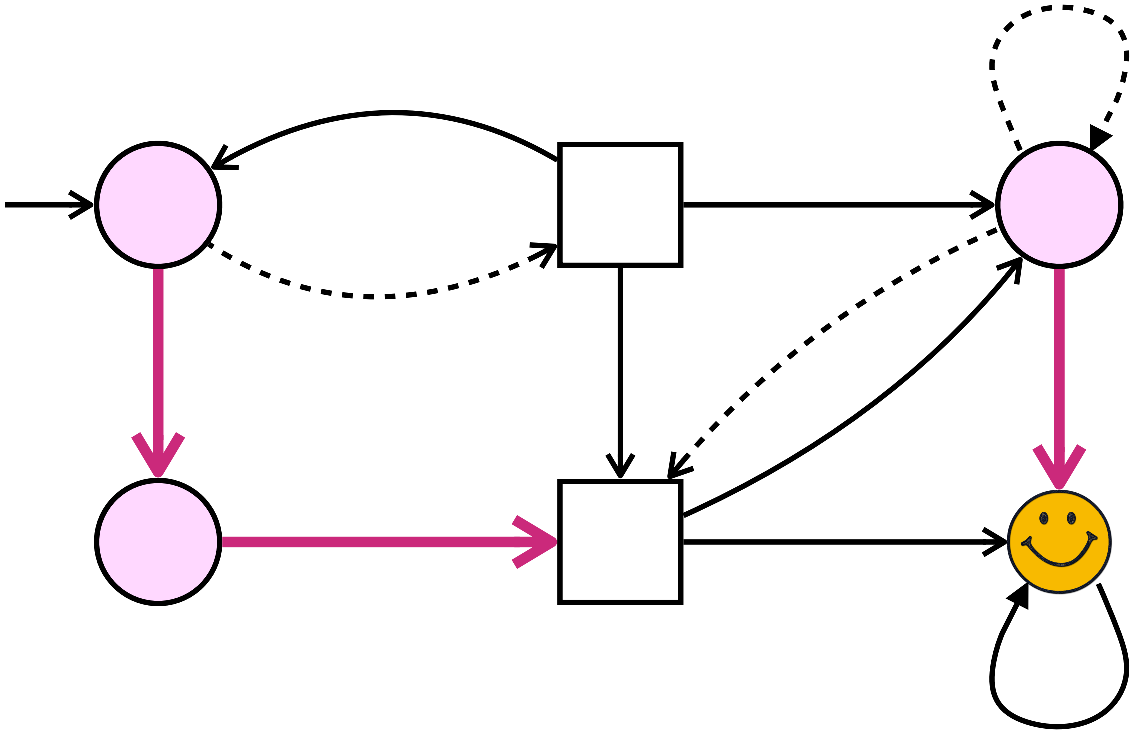
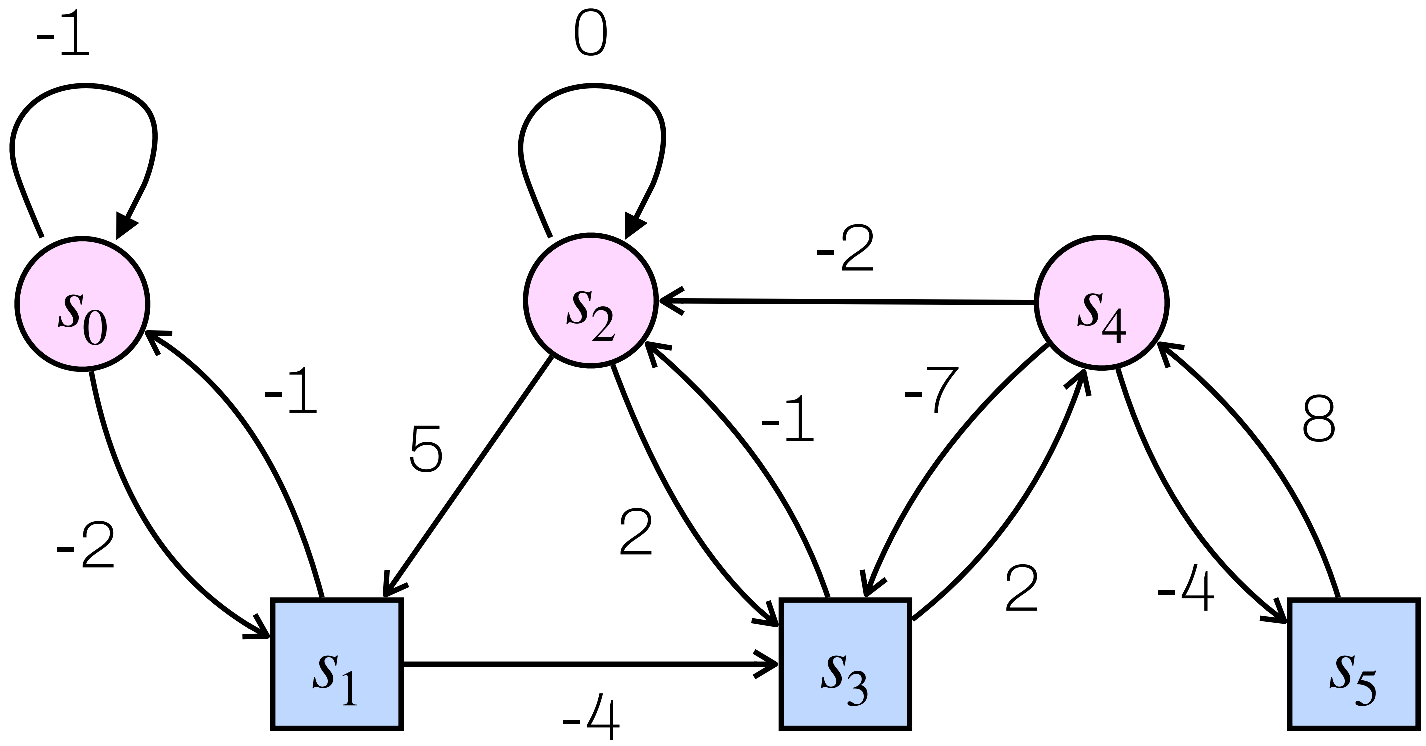


Example: mean-payoff

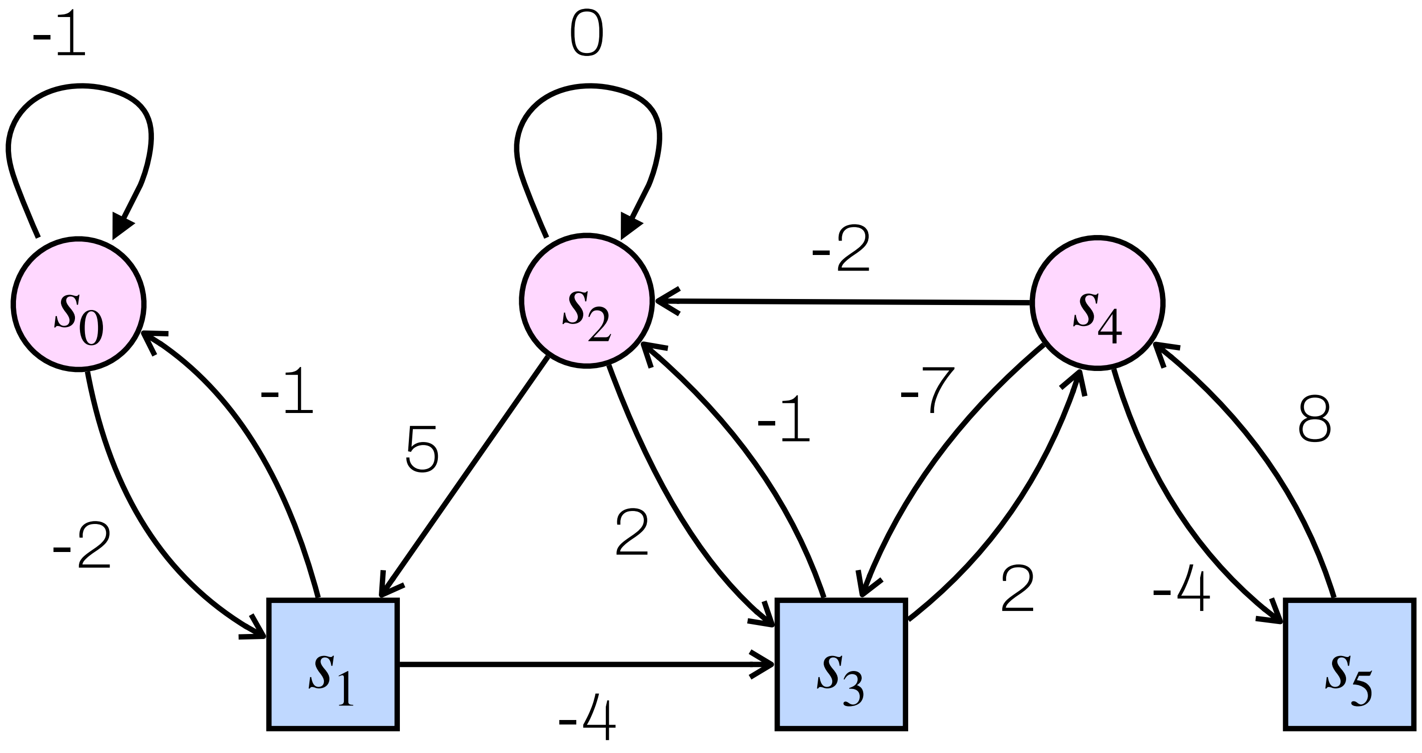


[Ohl21] Ohlmann. Monotonic graphs for Parity and Mean-Payoff games (PhD thesis).

Example: mean-payoff

- P_1 maximizes $\overline{\mathrm{MP}}$, P_2 minimizes $\overline{\mathrm{MP}}$

$$\overline{\mathrm{MP}} = \limsup_n \frac{\sum_{i \neq n} c_i}{n}$$



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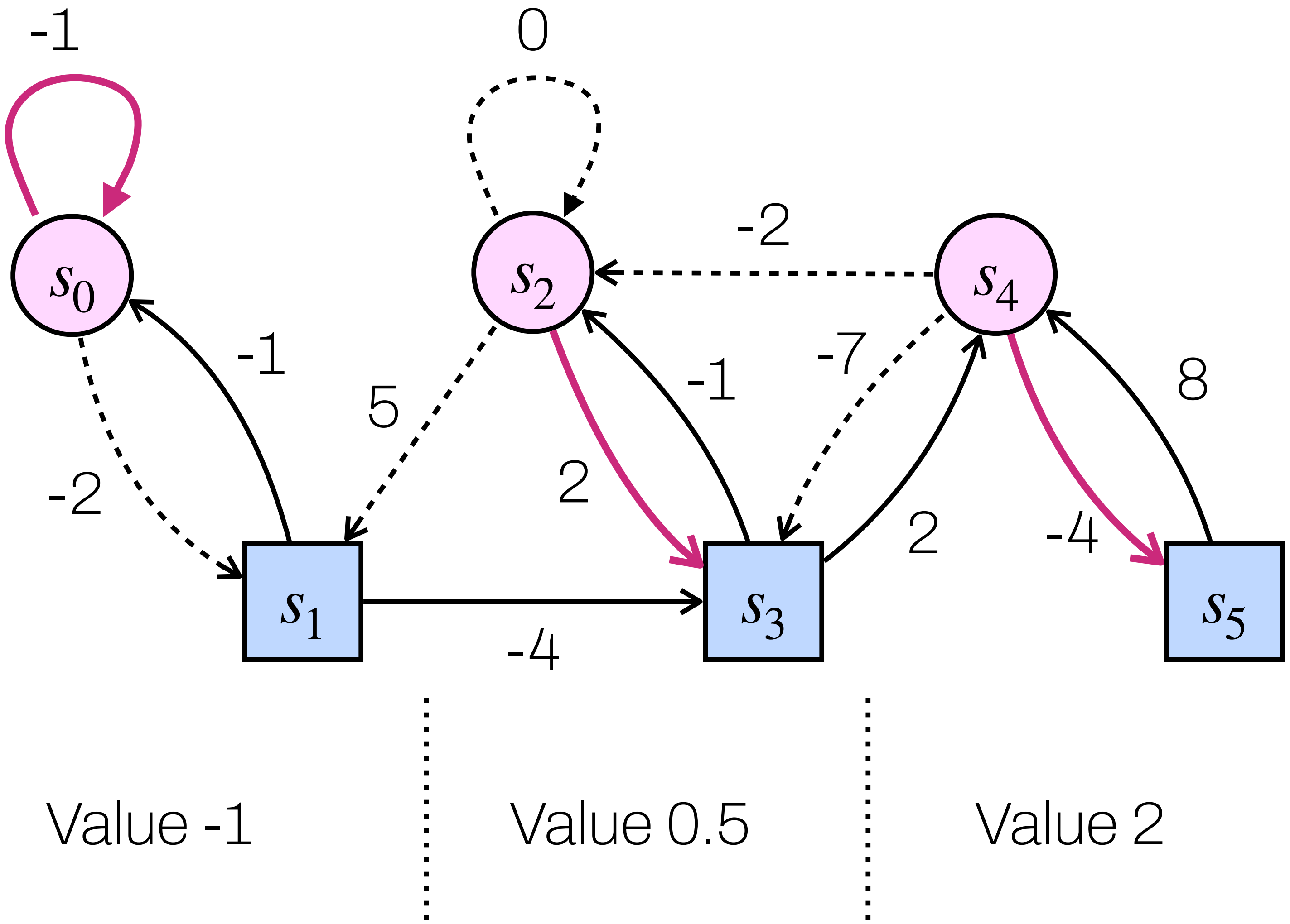

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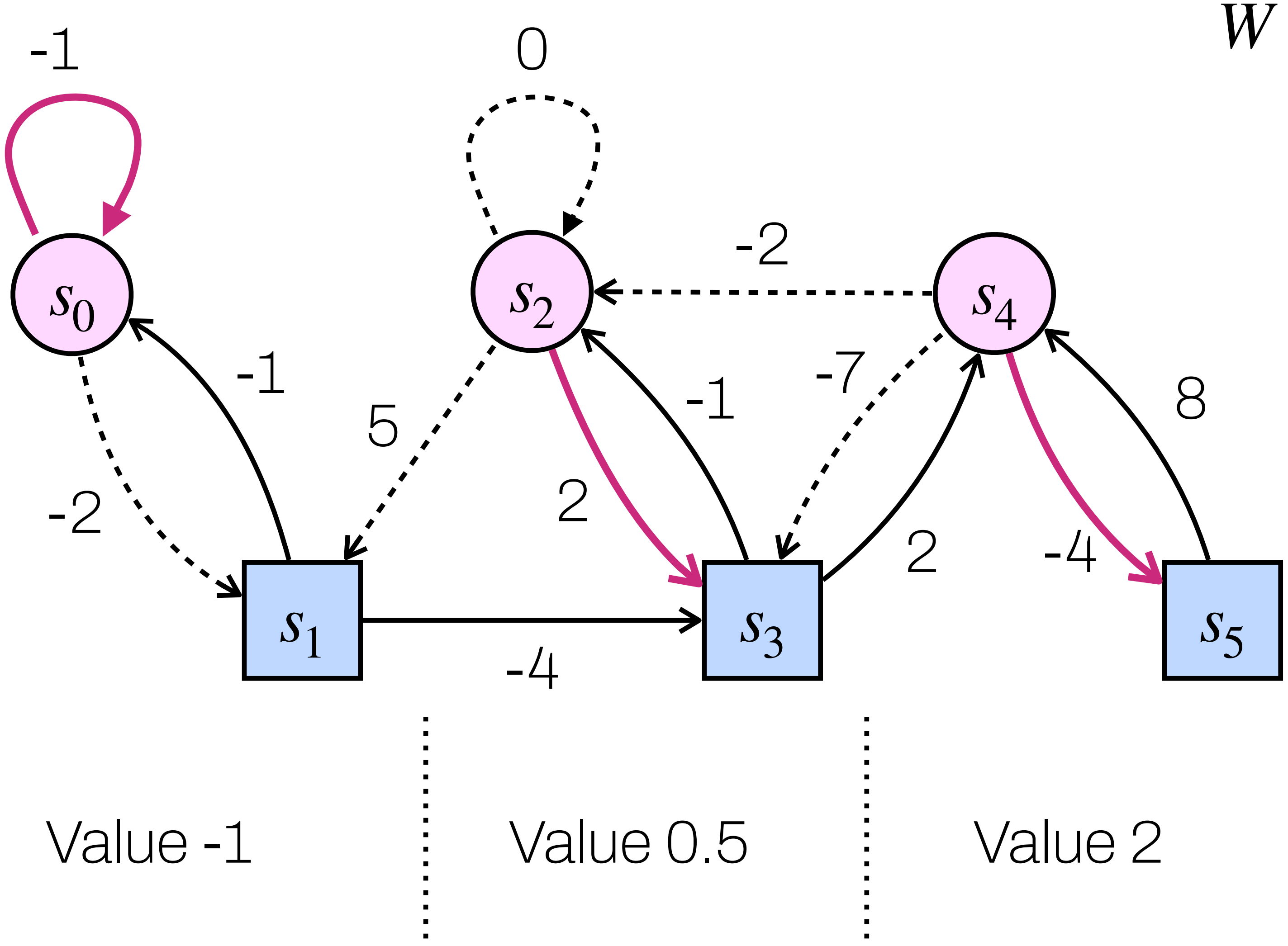

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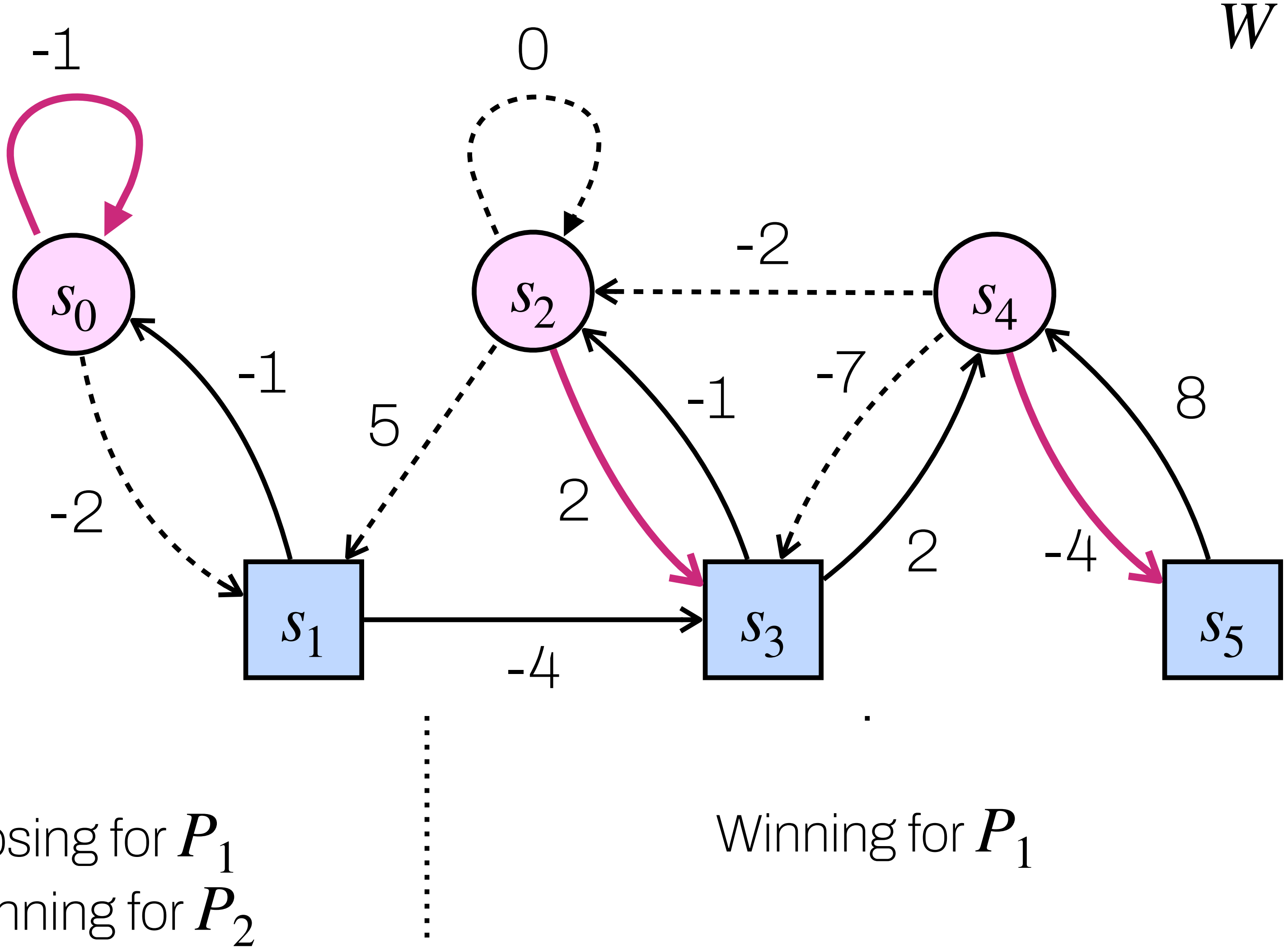

Losing for P_1
Winning for P_2

Winning for P_1

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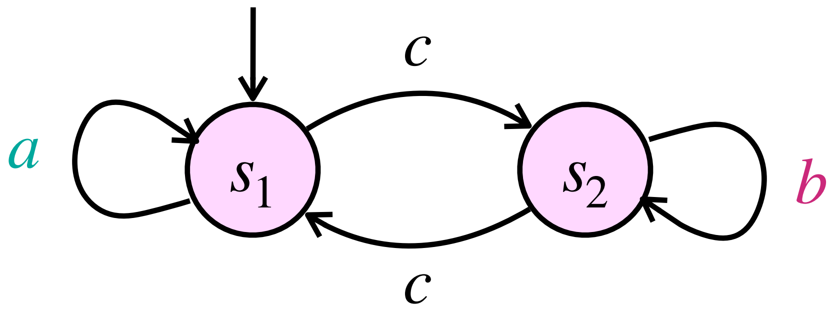
Do we need more?

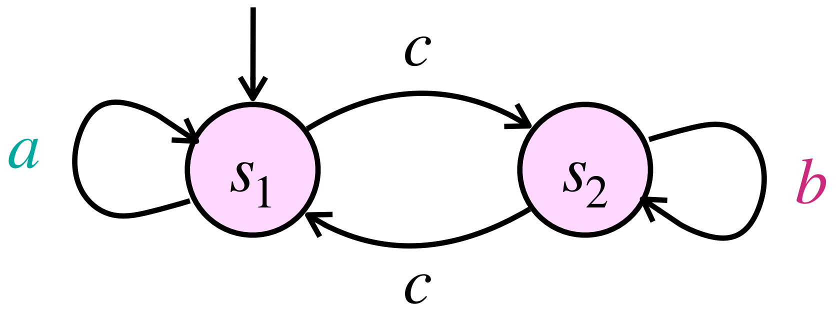
Examples



« See infinitely often both a and b »

Büchi(a) ∧ Büchi(b)

Examples



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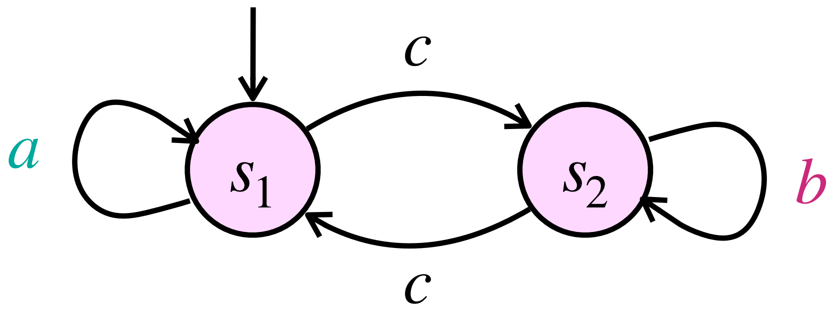
Winning strategy

- At each visit to s_1, loop once in s_1 and then go to s_2
- At each visit to s_2, loop once in s_2 and then go to s_1
- Generates the sequence $(acbc)^\omega$

Examples



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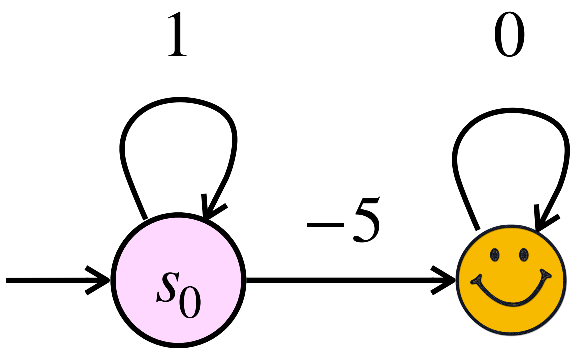

« Reach the target with energy level 0 »
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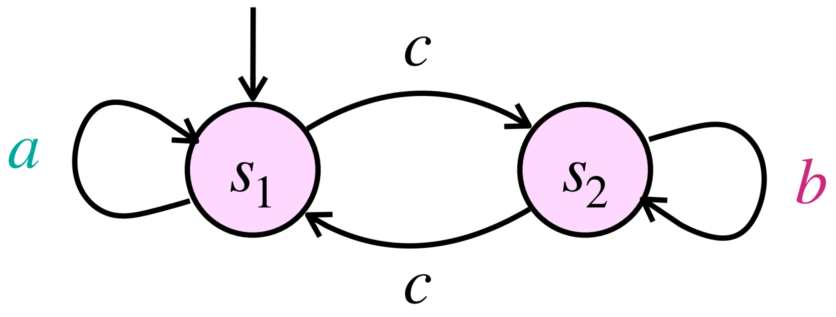
Examples



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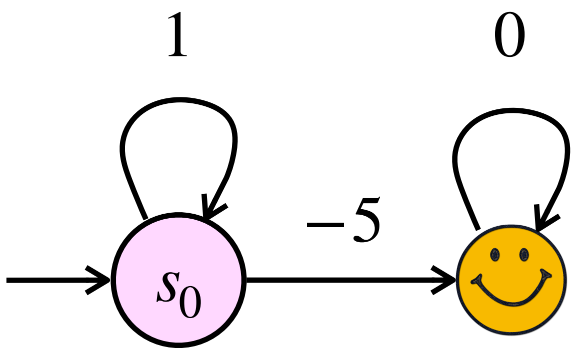

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- Loop five times in s_0
- Then go to the target
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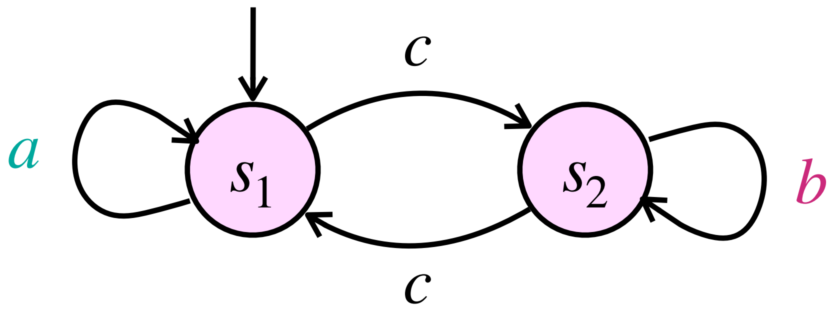
Examples



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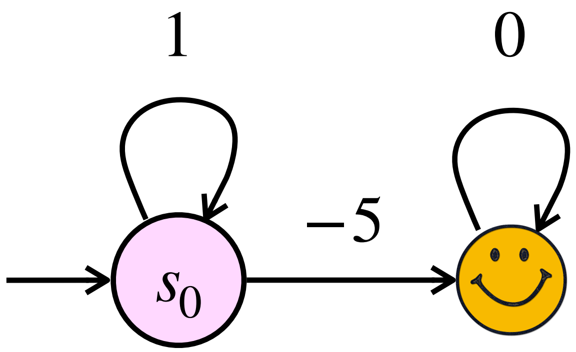

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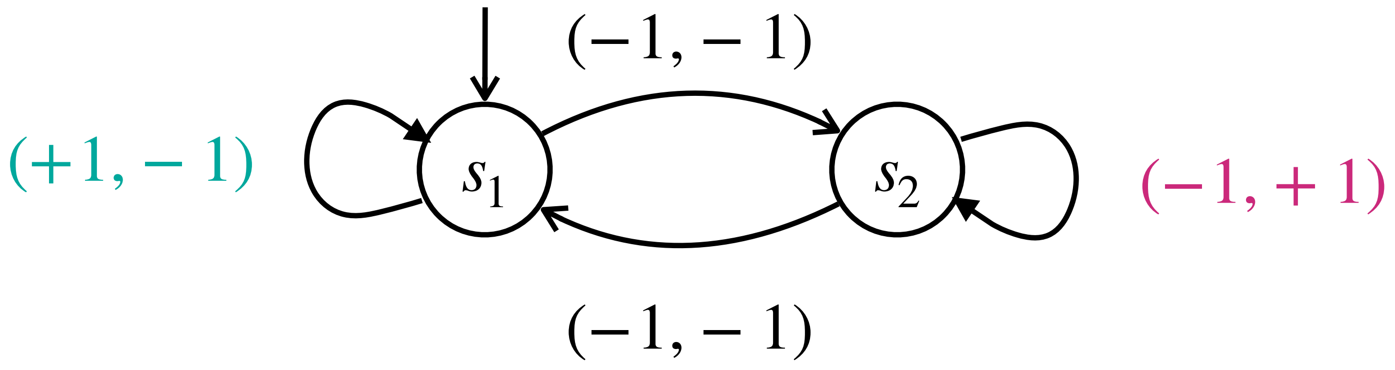
These two strategies require only **finite** memory

Example: multi-dimensional mean-payoff



« Have a (limsup) mean-payoff ≥ 0 on both dimensions »

So-called *multi-dimensional mean-payoff*

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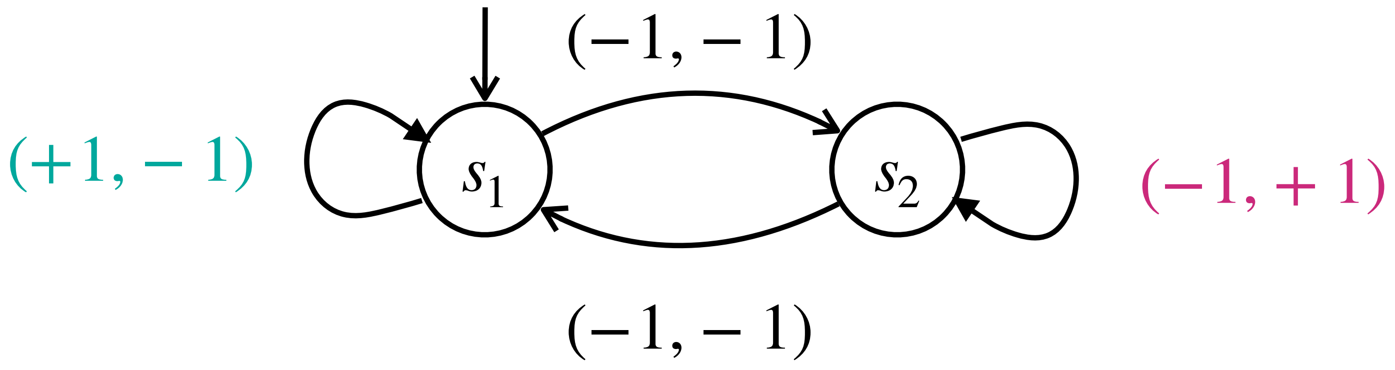

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- After k-th switch between s_1 and s_2, loop $2k-1$ times and then switch back
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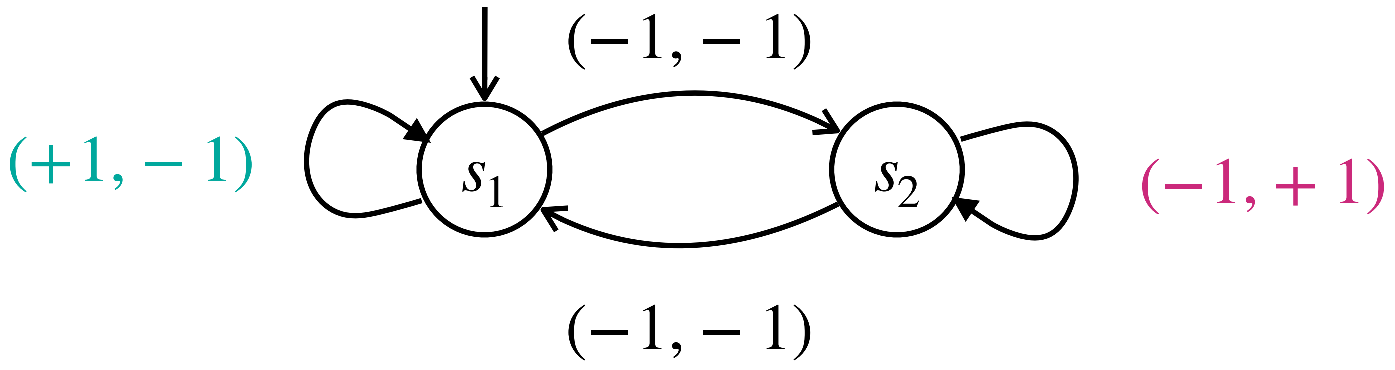

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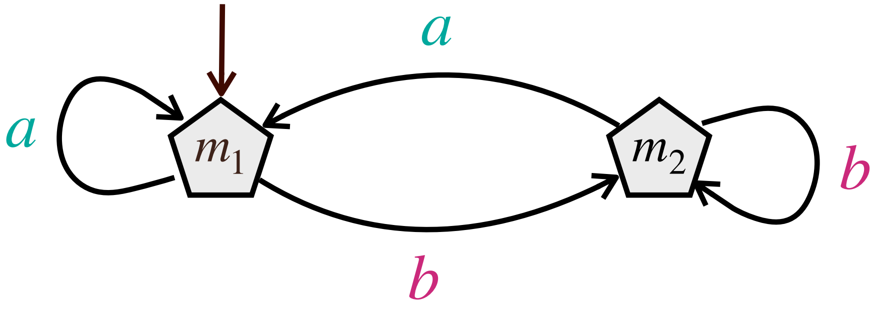
This strategy requires **infinite** memory, and this is unavoidable

We focus on finite memory!

Chromatic* memory

Memory skeleton

$\mathscr{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ with $m_{\text{init}} \in M$ and $\alpha_{\text{upd}} : M \times C \to M$



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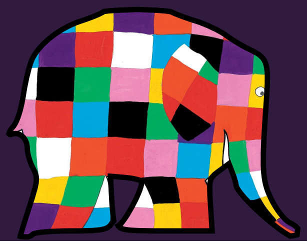
* Terminology by Kopczyński

Chromatic* memory



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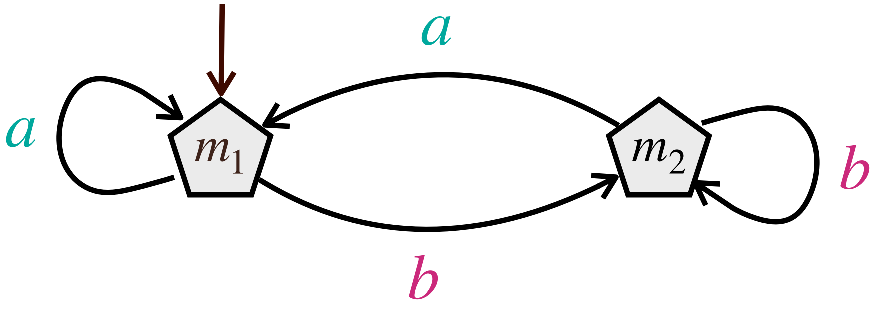

Not yet a strategy!

$\sigma_i : S^*S_i \to E$

* Terminology by Kopczyński

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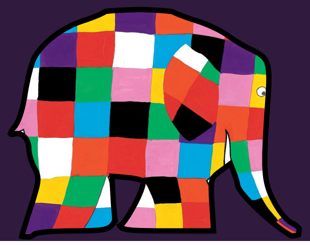

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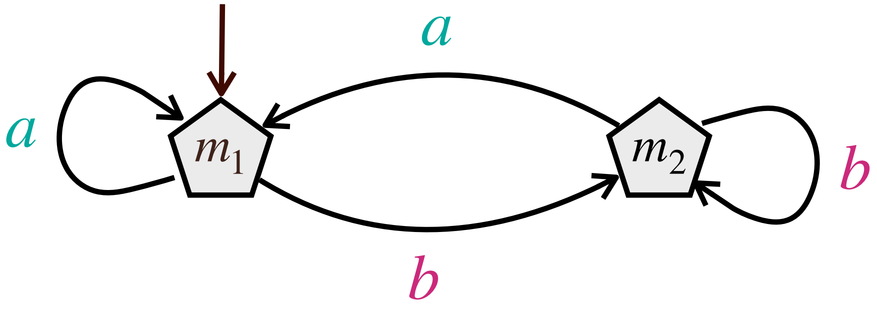


Not yet a strategy!

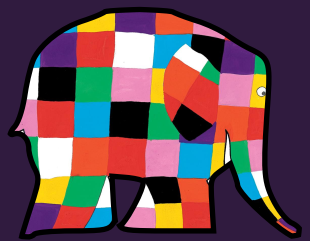
$\sigma_i : S^* S_i \to E$

Strategy with memory $\mathscr{M}$

Additional next-move function $\alpha_{\text{next}} : M \times S_i \to E$

$(\mathscr{M}, \alpha_{\text{next}})$ defines a strategy!

* Terminology by Kopczyński



Chromatic* memory

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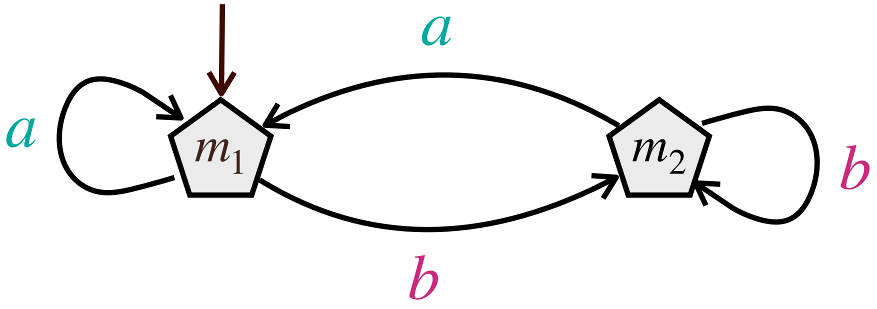

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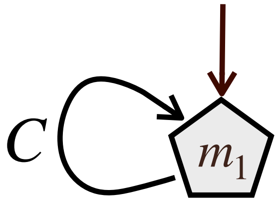

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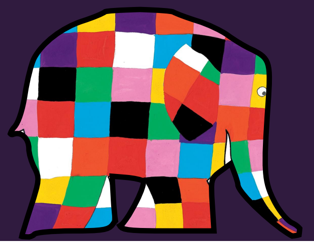

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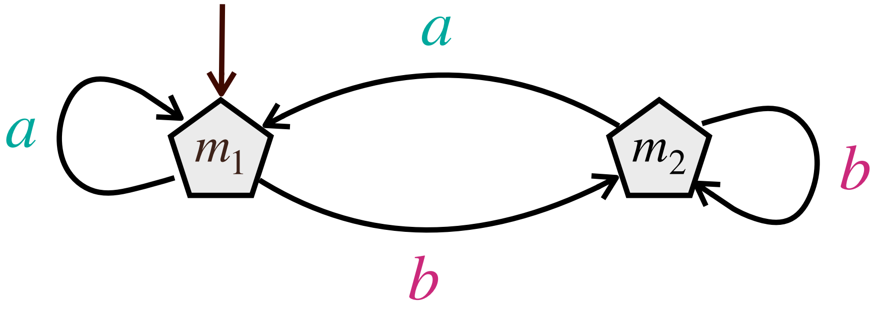

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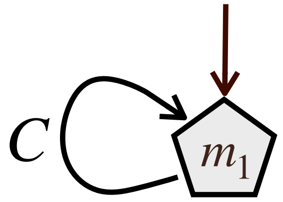

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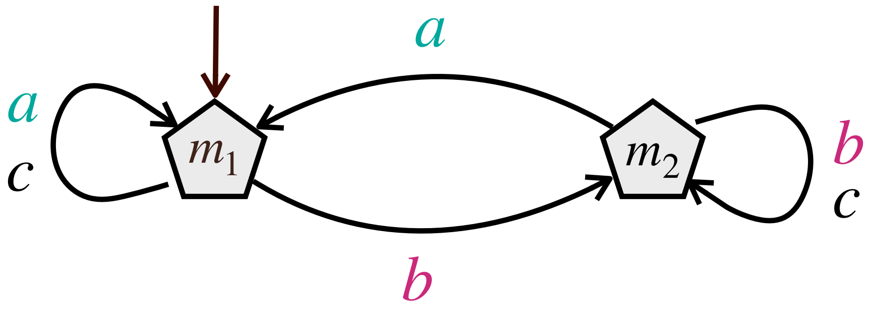
* Terminology by Kopczyński

Example of chromatic memory

$\mathscr{M}$



This skeleton is sufficient for winning

$W =$ Büchi$(a) \wedge$ Büchi(b) (in any arena)

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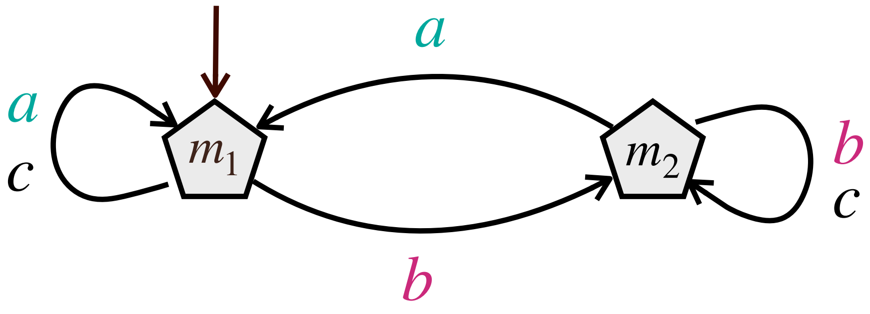

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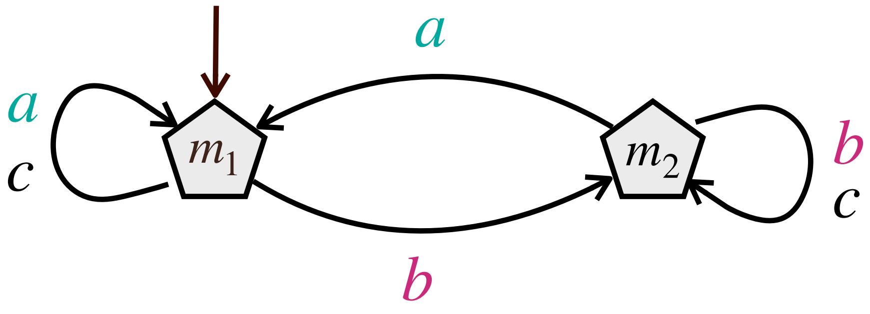

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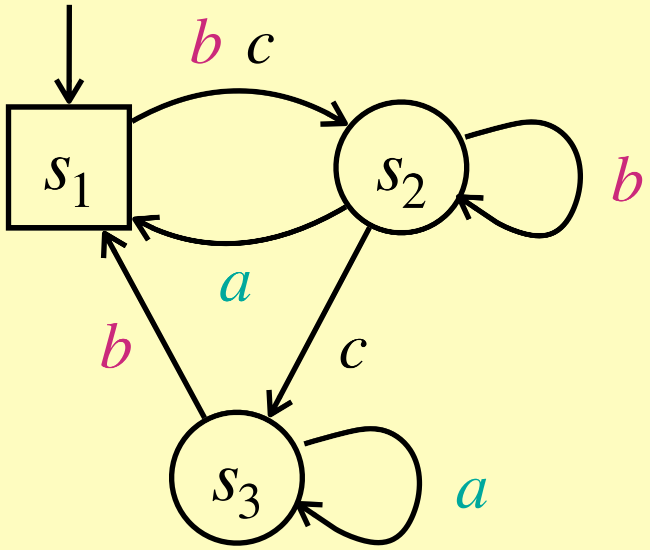

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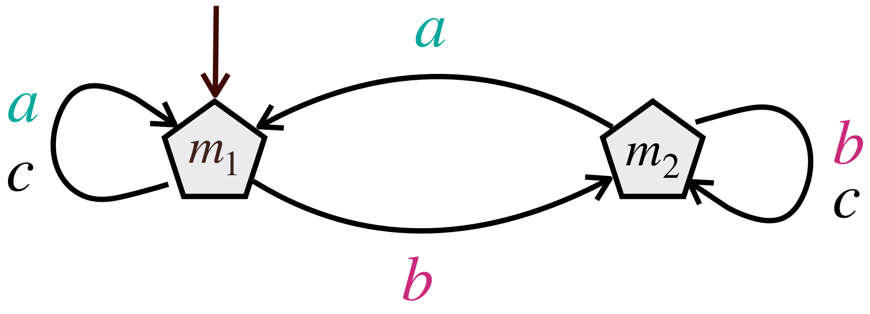

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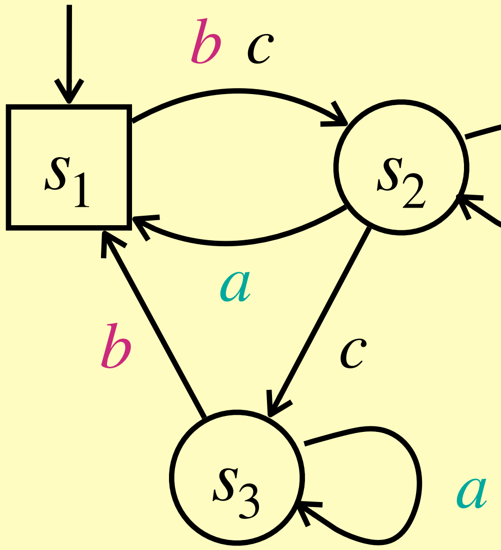

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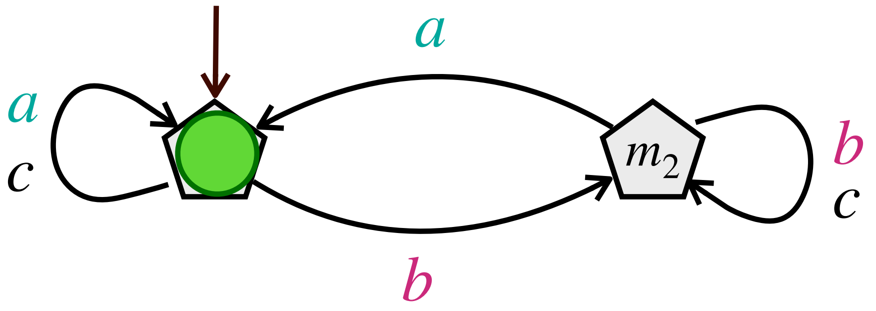

(m_1, s_1)

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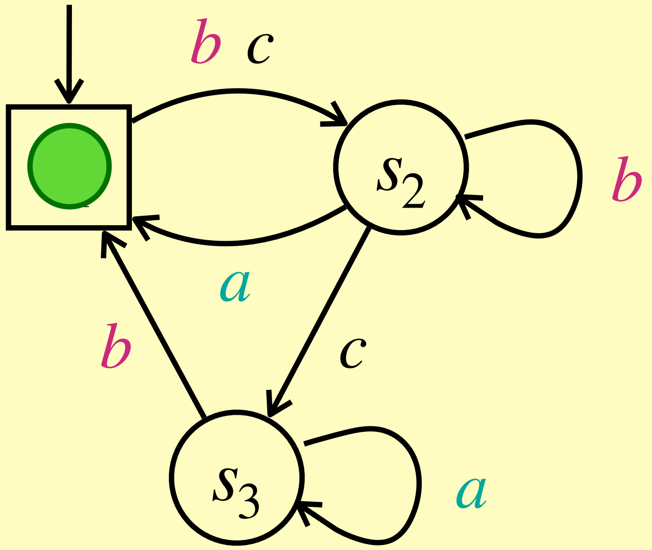
Example



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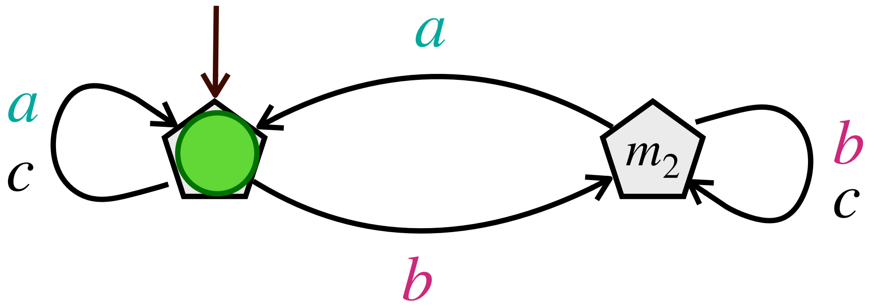
Example of chromatic memory

$\mathscr{M}$



$$(m_1, s_1) \xrightarrow{c} (m_1, s_2)$$

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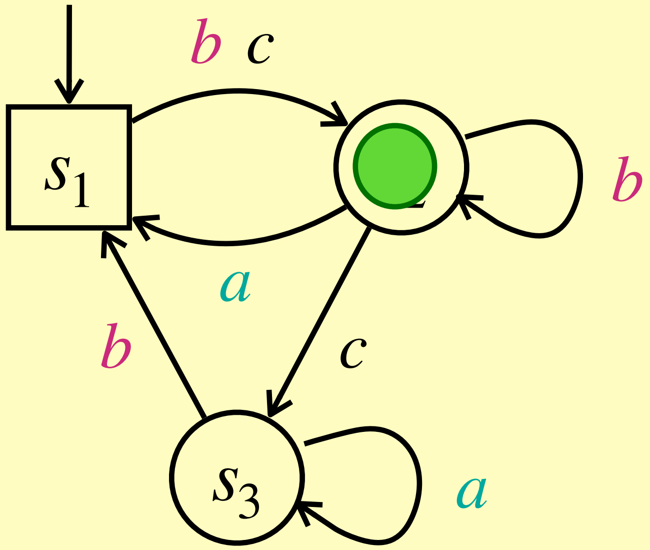
Example



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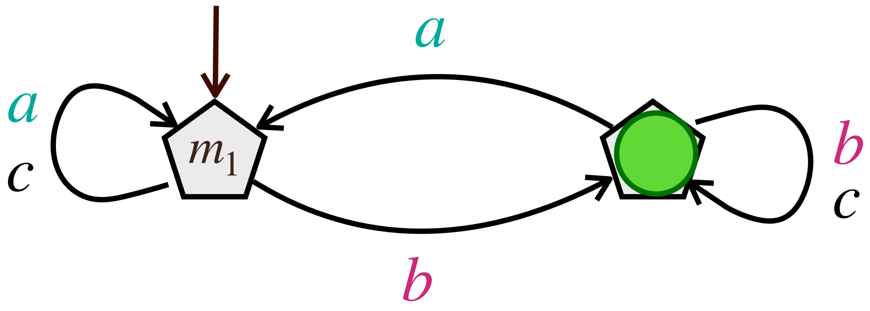
Example of chromatic memory

$\mathscr{M}$



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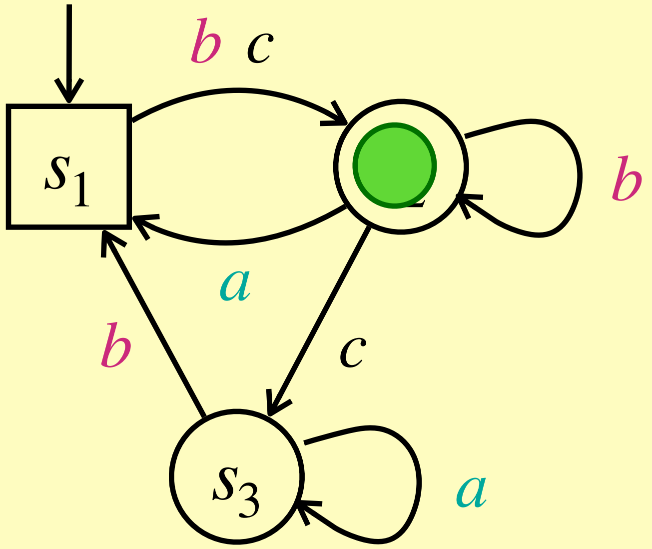

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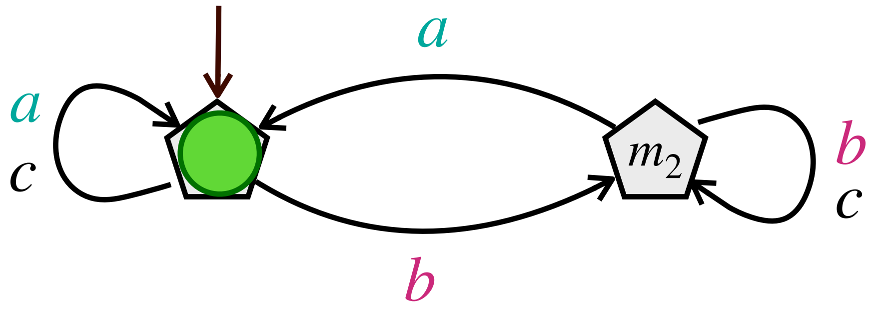

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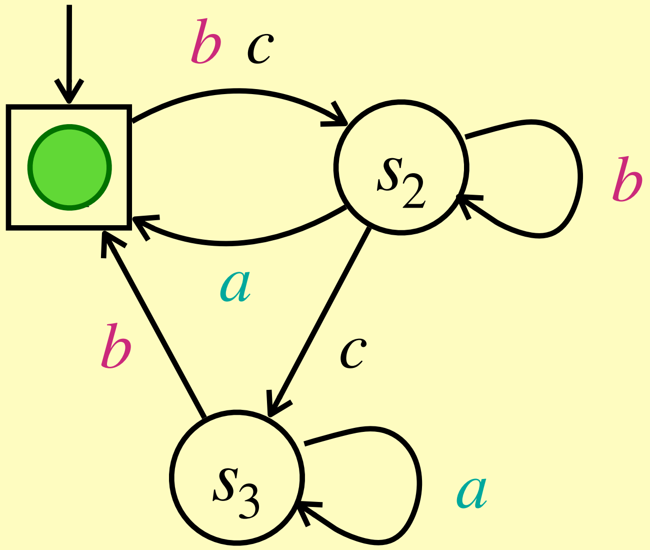

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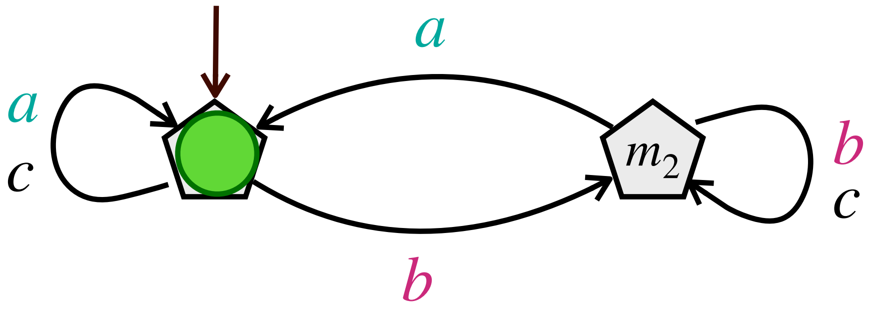

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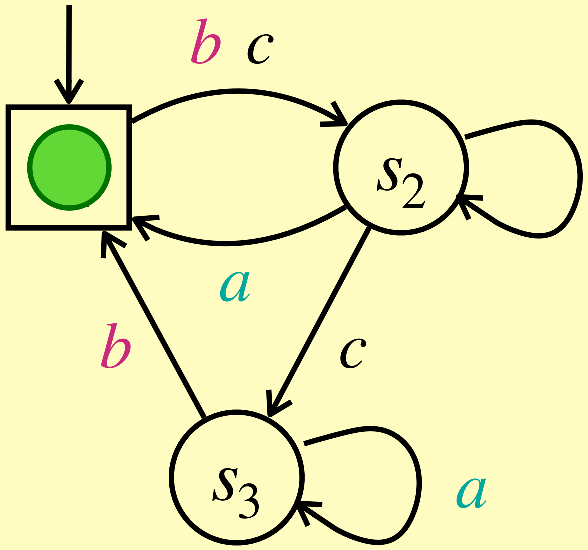

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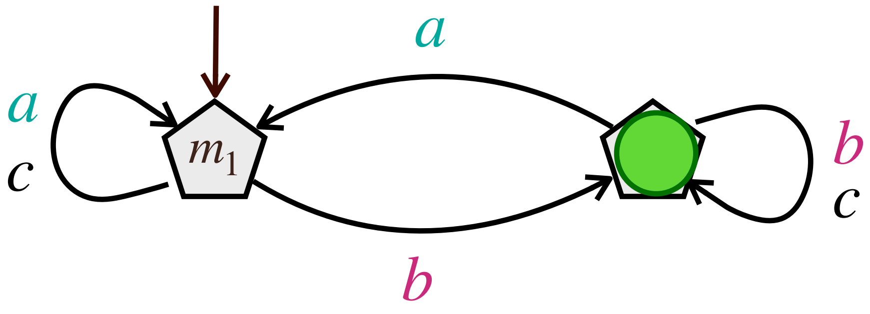

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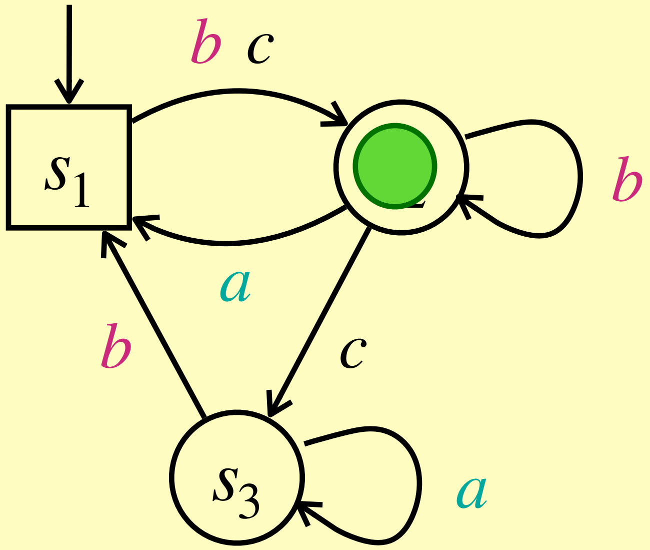

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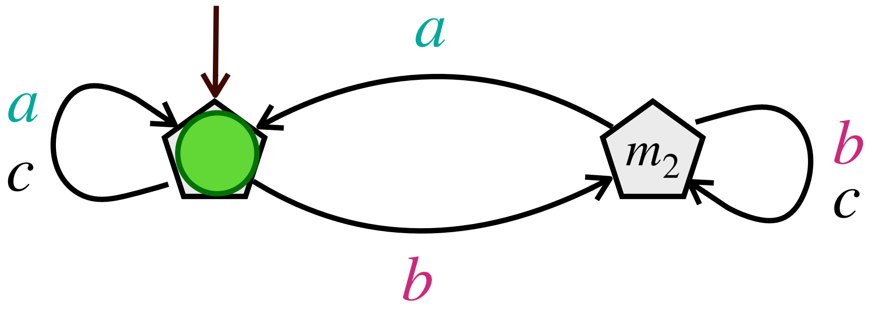

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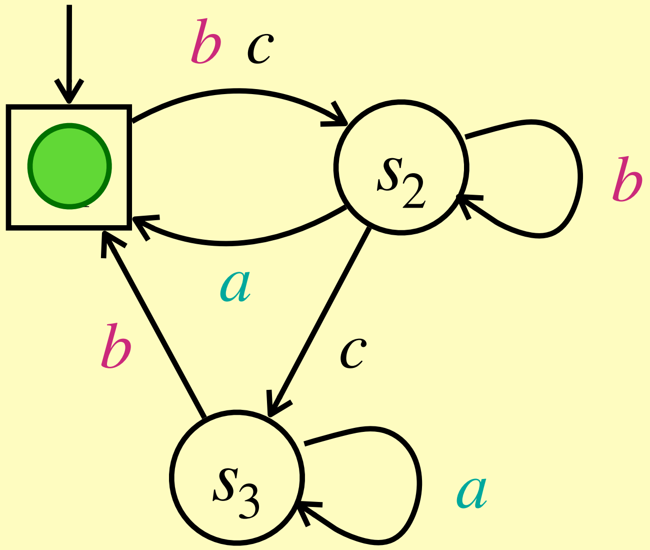

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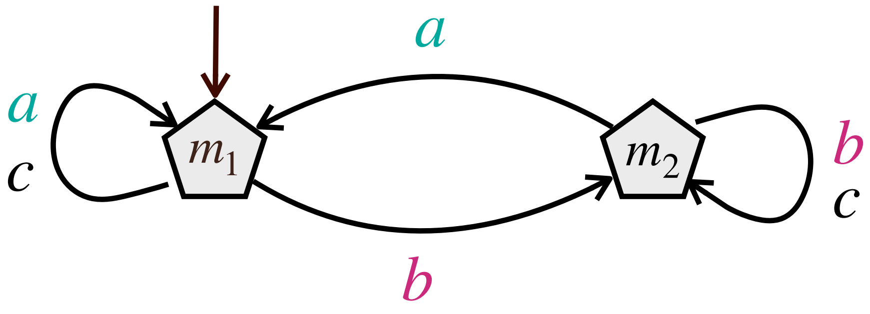

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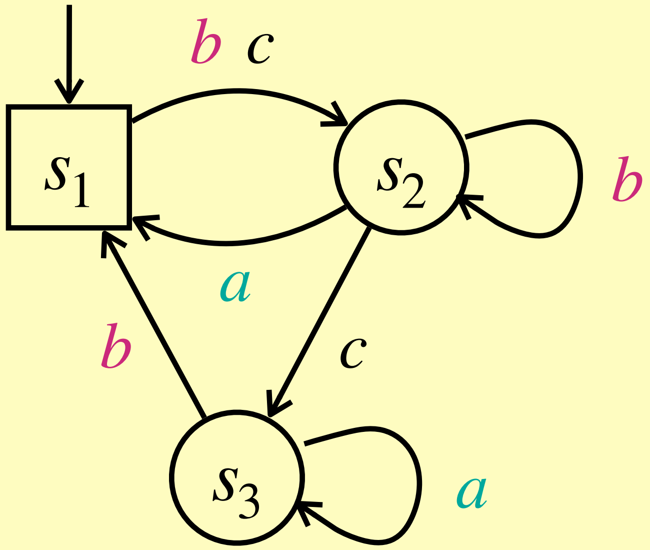

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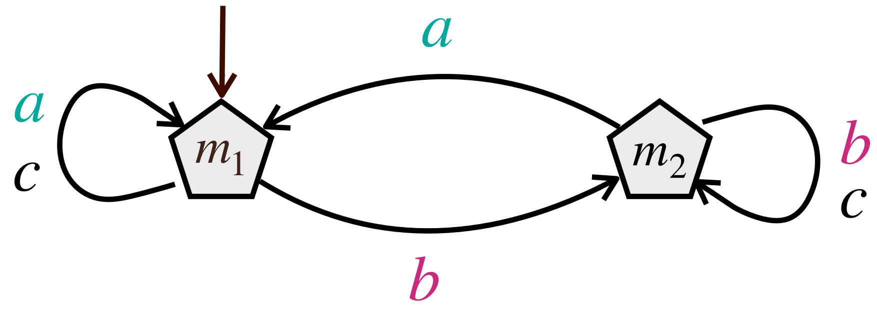
Example of chromatic memory

$\mathcal{M}$



$$(m_1, s_1) \xrightarrow{c} (m_1, s_2) \xrightarrow{b} (m_2, s_2) \xrightarrow{a} (m_1, s_1)$$

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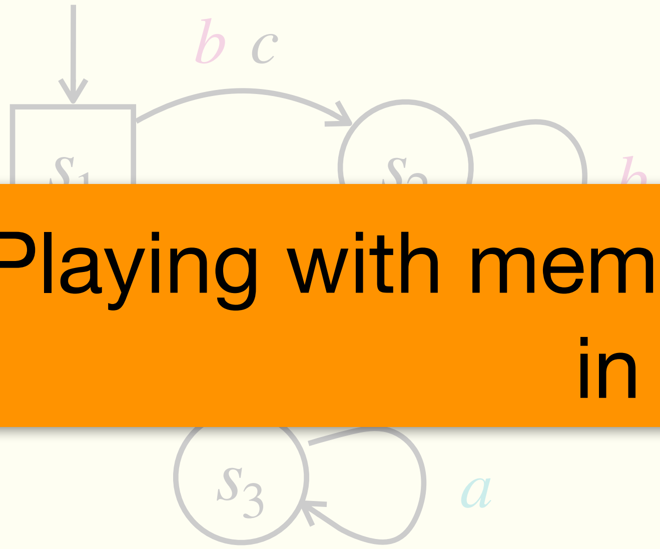

$\alpha_{\text{next}} : M \times S_1 \to E$

Playing with memory $\mathcal{M}$ is like playing memoryless in the product arena

$(m_\star, s_3) \mapsto (s_3, b, s_1)$

A zoology of notions

A zoology of notions

- Let W be an objective and $i \in \{1,2\}$

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- A skeleton $\mathscr{M}$ suffices to win for P_1 (resp. P_2) for W if P_1 (resp. P_2) has an optimal* strategy based on $\mathscr{M}$ in any game $(\mathscr{A}, W)$ (resp. $(\mathscr{A}, W^c)$)

* That is, it is winning whenever it is possible to win

A zoology of notions

- Let W be an objective and $i \in \{1,2\}$

in finite arenas
in one-player arenas

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- W is $\mathscr{M}$-determined if $\mathscr{M}$ suffices to win for both players for W

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in finite arenas
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- W is $\mathscr{M}$-determined if $\mathscr{M}$ suffices to win for both players for W
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- Finite-memory determined = $\exists \mathscr{M}$ s.t. $\mathscr{M}$-determined

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in finite arenas
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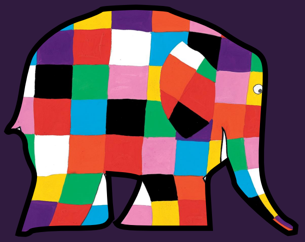
finite
one-player

- W is $\mathscr{M}$-determined if $\mathscr{M}$ suffices to win for both players for W
- Memoryless determined = $\mathscr{M}_{\text{triv}}$-determined
- Finite-memory determined = $\exists \mathscr{M}$ s.t. $\mathscr{M}$-determined
- W is half-positional = $\mathscr{M}_{\text{triv}}$ suffices to play optimally for P_1 for W

---

* That is, it is winning whenever it is possible to win

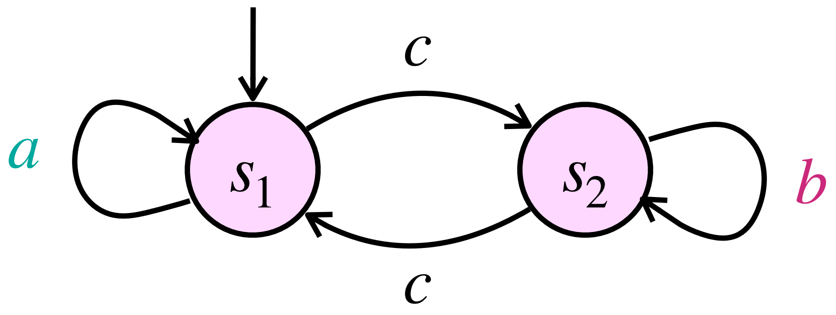
Warning



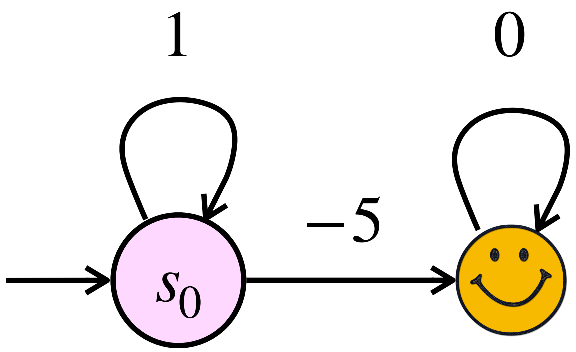
$\mathscr{M}$-determinacy requires

- Chromatic memory: the skeleton is based on colors
- Arena-independent memory: the same memory skeleton is used in all arenas (of the designed class)

Examples



« See infinitely often both a and b »
Büchi(a) ∧ Büchi(b)



« Reach the target with energy level 0 »
$\mathbf{FG}\ (\text{EL} = 0)$

Winning strategy

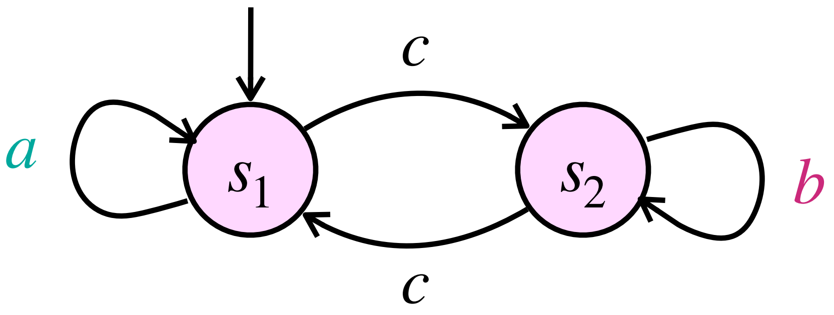
- At each visit to s_1, loop once in s_1 and then go to s_2
- At each visit to s_2, loop once in s_2 and then go to s_1
- Generates the sequence $(acbc)^\omega$

Winning strategy

- Loop five times in s_0
- Then go to the target
- Generates the sequence of colors $1\ 1\ 1\ 1\ 1\ \ -5\ 0\ 0\ 0...$

These two strategies require only **finite** memory

Examples



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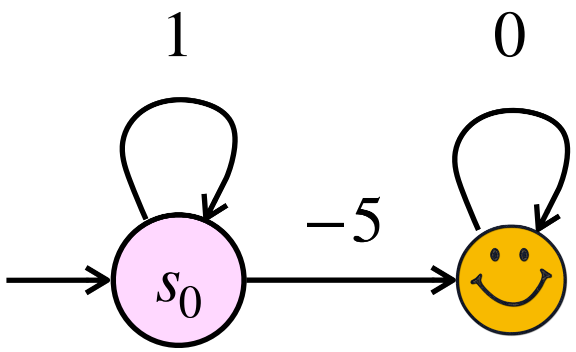

« Reach the target with energy level 0 »

$\mathbf{FG}$ (EL $= 0$)

Winning strategy

- At each visit to s_1, loop once in s_1 and then go to s_2
- Generates the sequence $(acbc)^\omega$

There is an arena-independent memory based on a skeleton

Winning strategy

- Loop five times in s_0

$1\ 1\ 1\ 1\ 1\ -5\ 0\ 0\ 0\ldots$

The memory has to be arena-dependent

These two strategies require only **finite** memory

Our goal

Understand well low-memory specifications

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Understand well low-memory specifications

Memoryless / finite-memory determinacy

Is it the case that memoryless (resp. finite-memory) strategies suffice to win when winning strategies exist?

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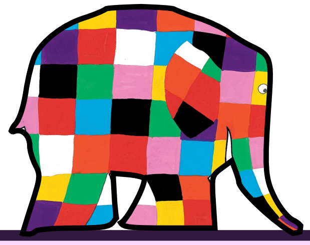

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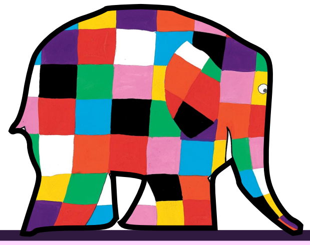

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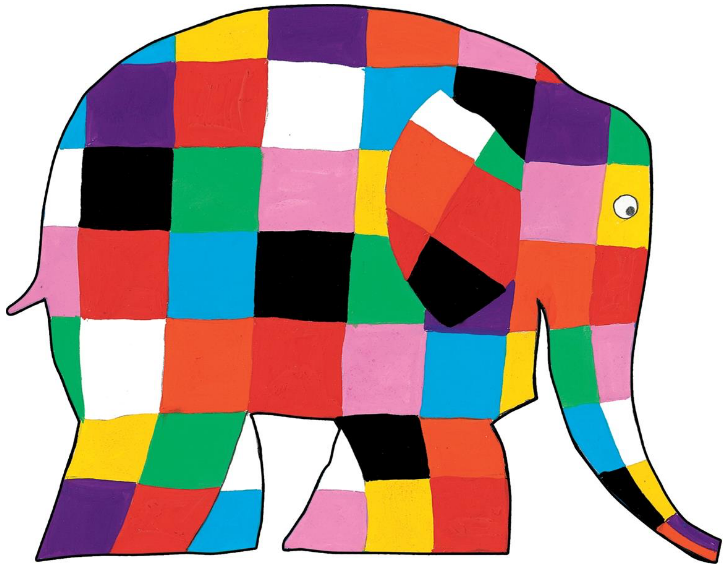
- Finite vs infinite games

Characterizing positional and chromatic finite-memory determinacy in finite games



A fundamental reference: [GZ05]

Sufficient conditions

- Sufficient conditions to guarantee memoryless optimal strategies for both players [GZ04,AR17]
- Sufficient conditions to guarantee half-positional optimal strategies [Kop06,Gim07,GK14]

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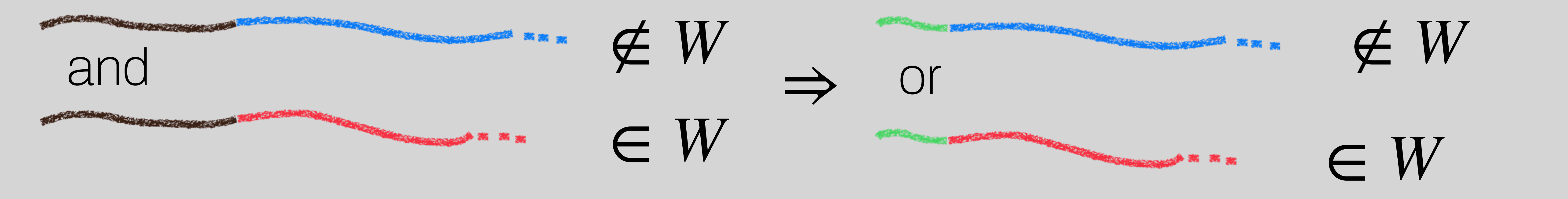

$\notin W$

and

$\in W$

$\Rightarrow$

$\notin W$

or

$\in W$

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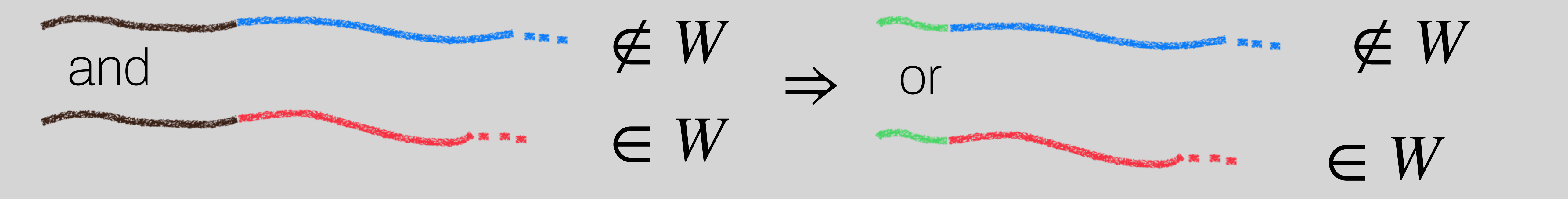
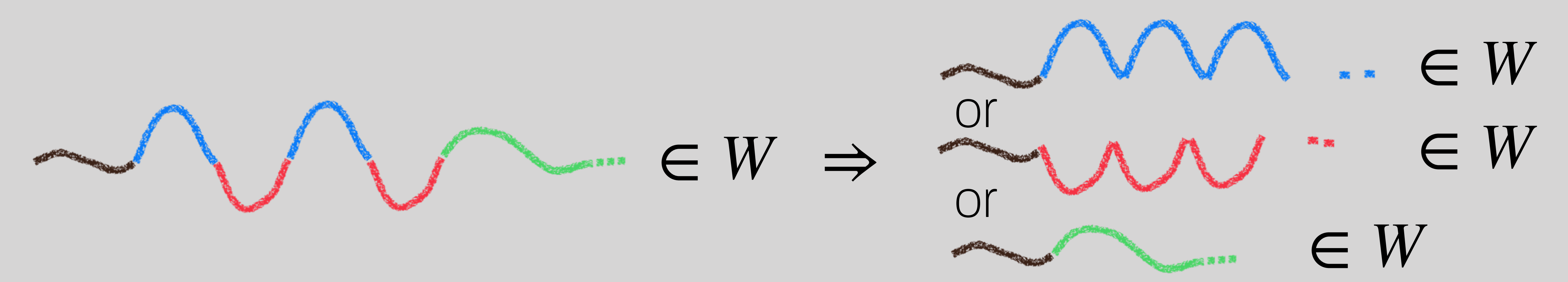


$\notin W$ and $\in W$ $\Rightarrow$ $\notin W$ or $\in W$

- W is **selective** whenever:



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Let W be an objective

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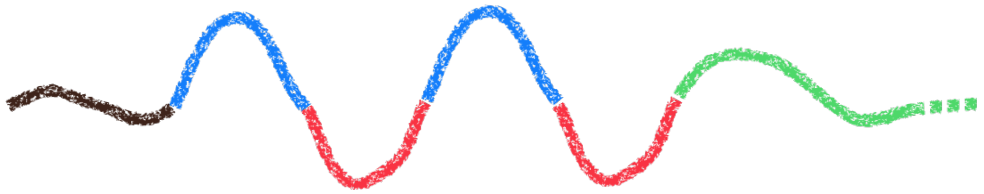

is winning

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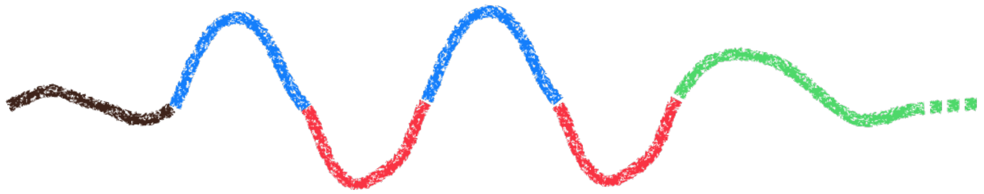
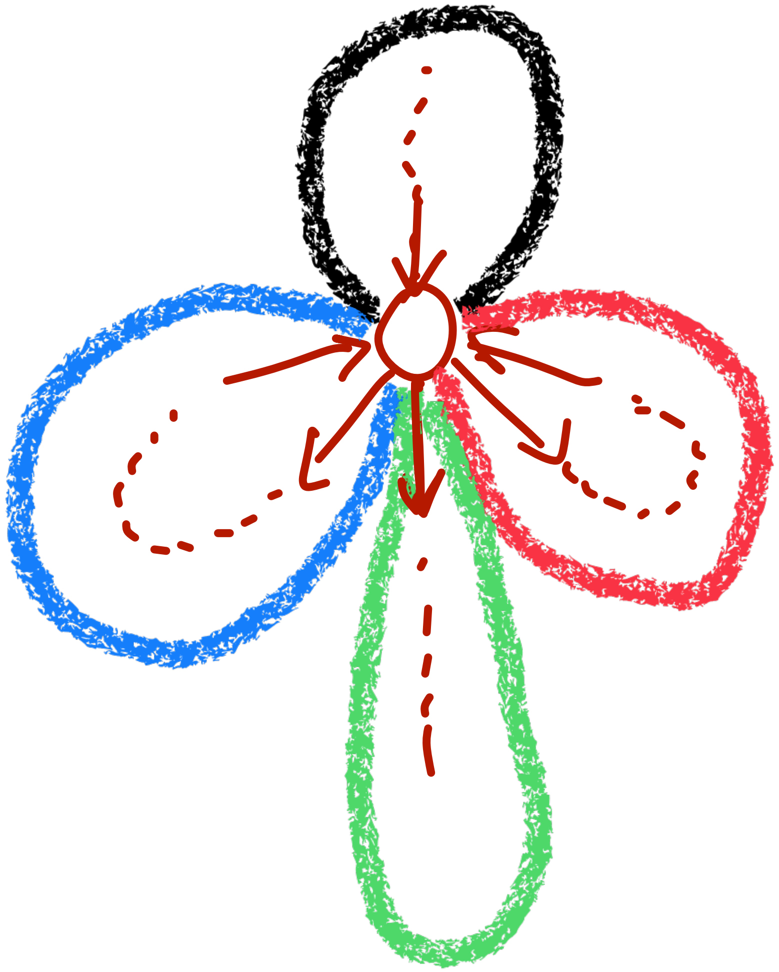
If  is winning



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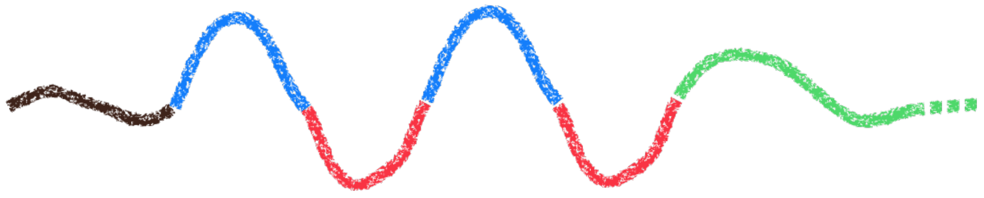
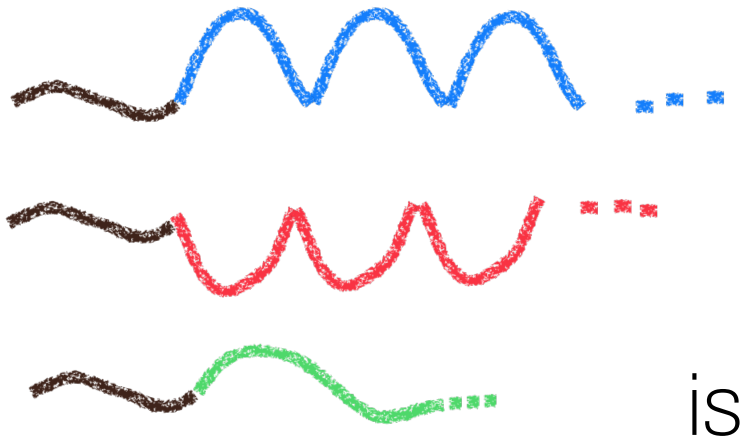


is winning

Then one of



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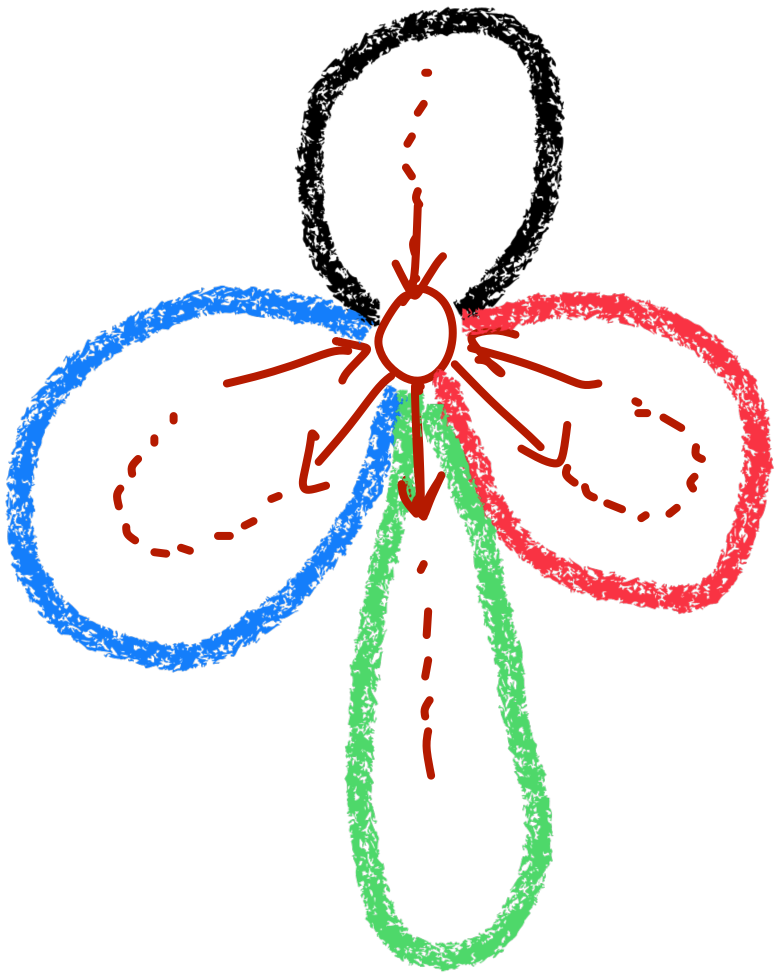

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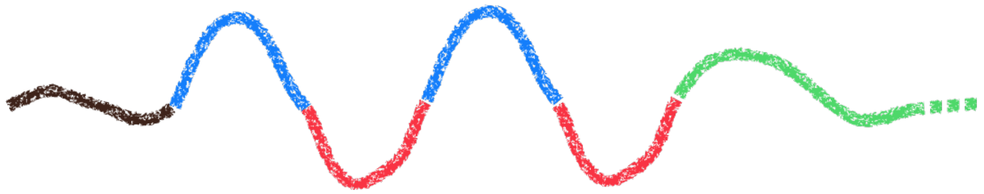

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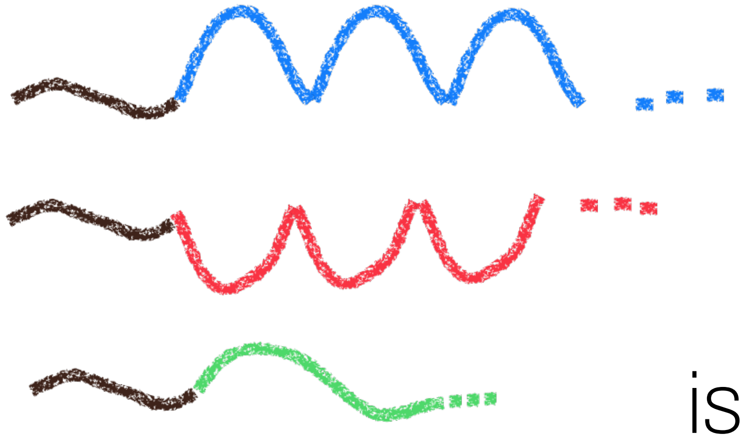

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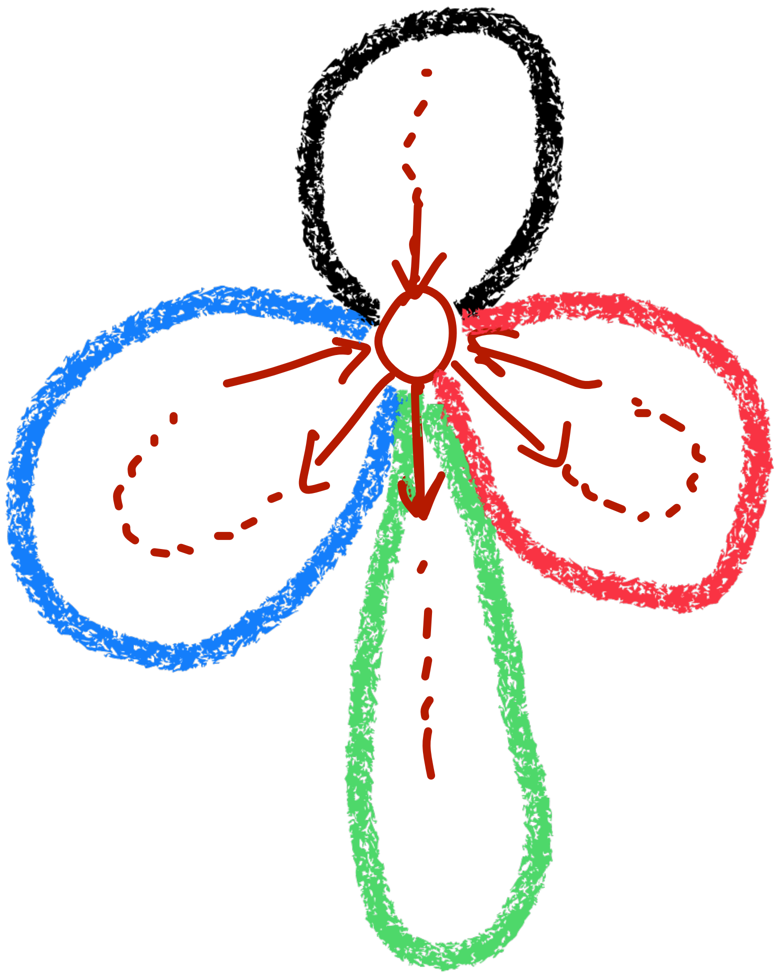

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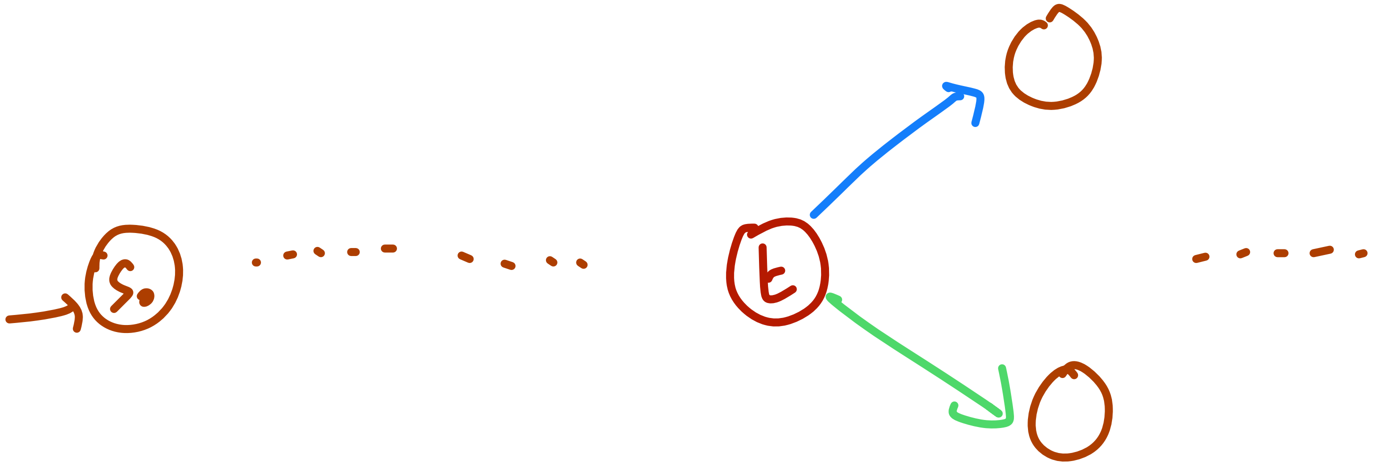

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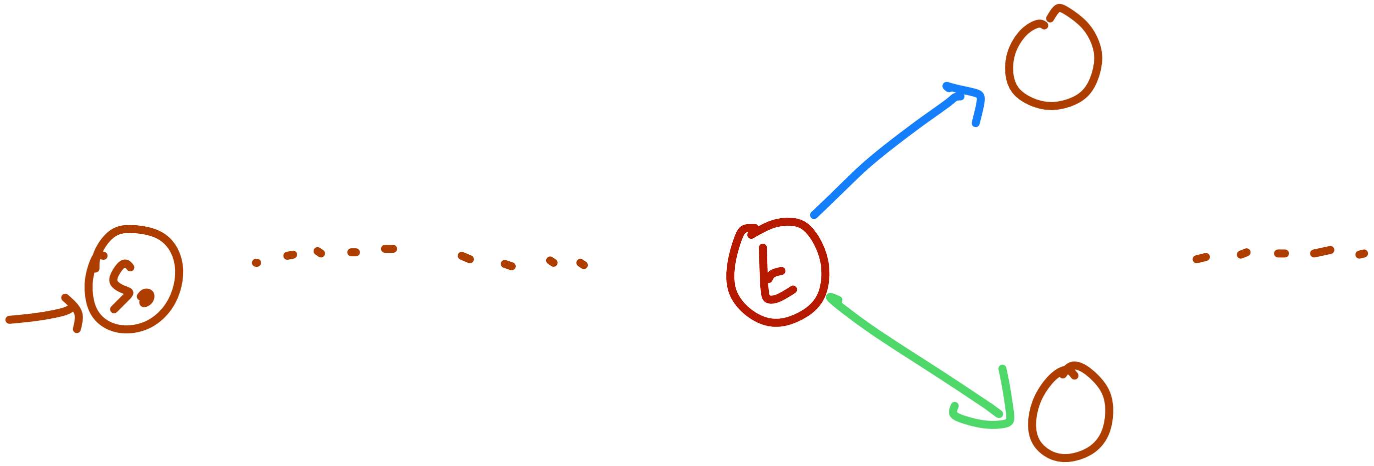

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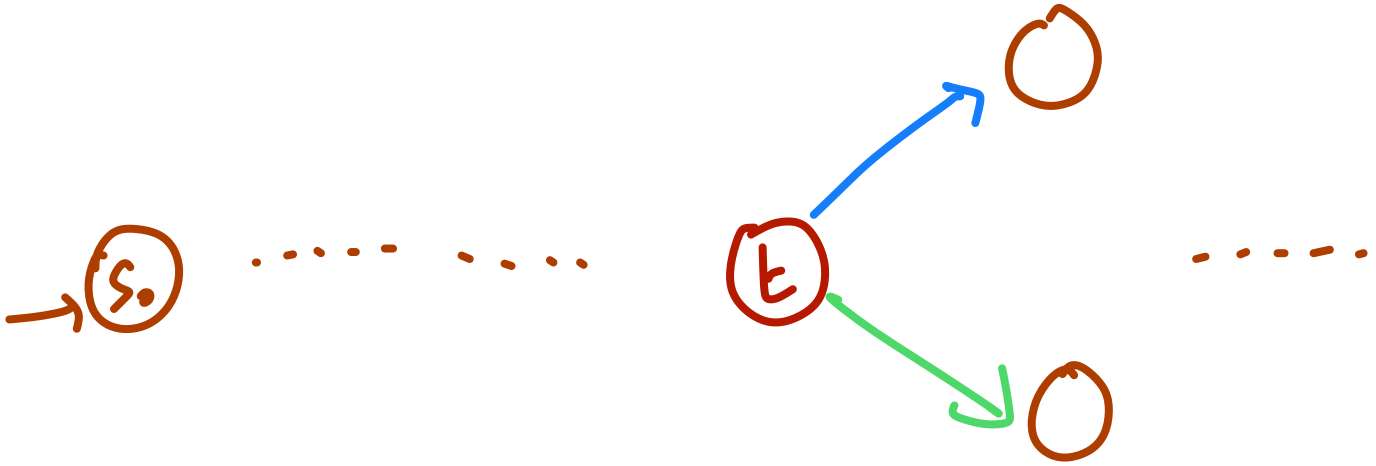

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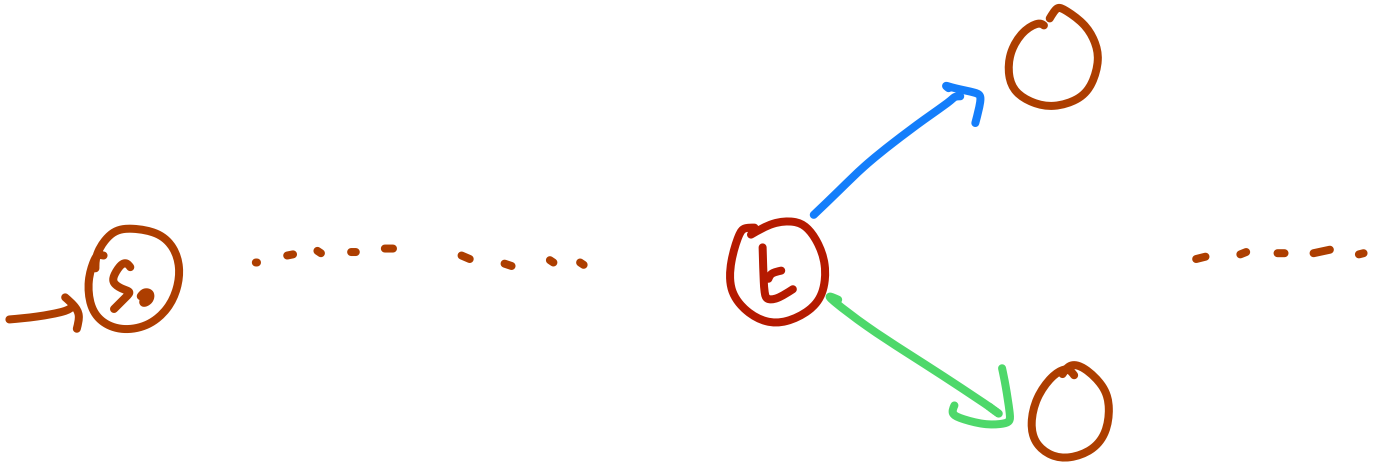

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No memory required at t!

Two characterizations

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Applications

Lifting theorem

Memoryless strategies suffice for W for P_i $(i = 1{,}2)$ in finite P_i-arenas

$$\Downarrow$$

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Very powerful and extremely useful in practice

- Easy to analyse the one-player case (graph reasoning)
  - Mean-payoff, average-energy [BMRLL15]
- Lift to two-player games via the theorem

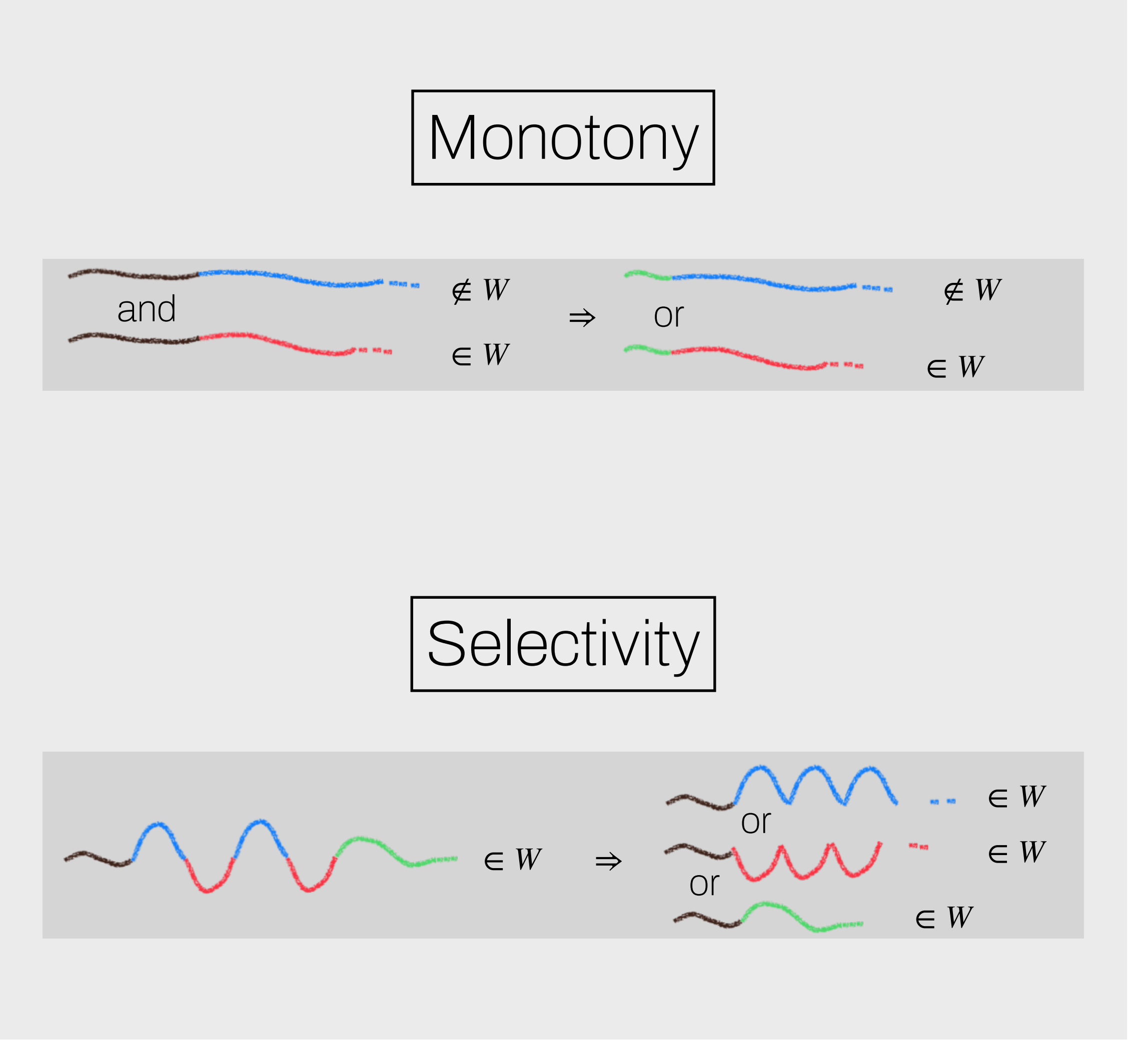
[BMRLL15] Bouyer, Markey, Randour, Larsen, Laursen. Average-energy games (GandALF'15).

Discussion of examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective

- Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective

- Average-energy games [BMRLL15]
  - Lifting theorem!!



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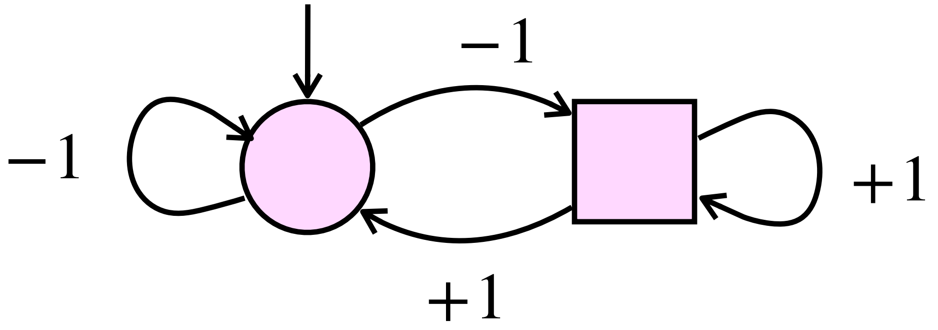

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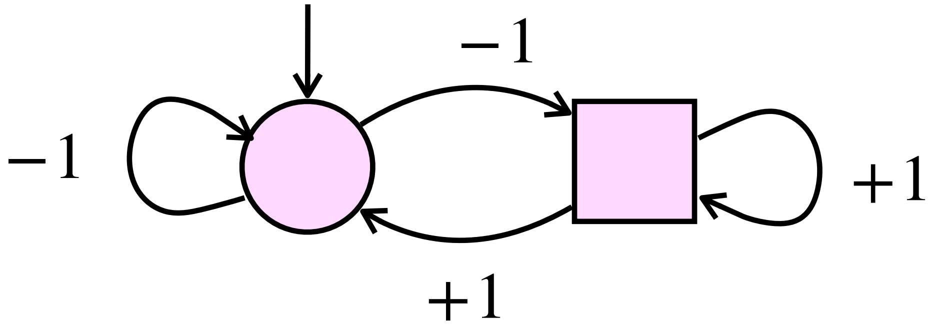
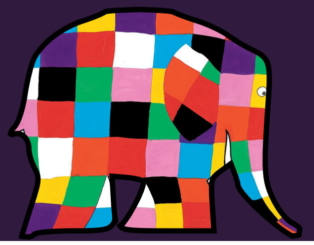


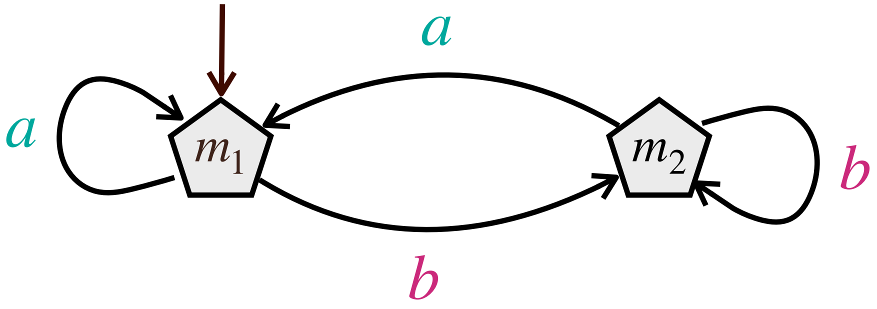
P_1 wins but requires infinite memory



Chromatic memory

Memory skeleton

$$\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}}) \text{ with } m_{\text{init}} \in M \text{ and } \alpha_{\text{upd}} : M \times C \to M$$



Not yet a strategy!

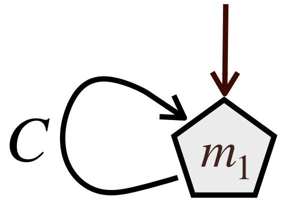
$\sigma_i : S^*S_i \to E$

Strategy with memory $\mathcal{M}$

Additional next-move function $\alpha_{\text{next}} : M \times S_i \to E$

$(\mathcal{M}, \alpha_{\text{next}})$ defines a strategy!

Remark: memoryless strategies are $\mathcal{M}_{\text{triv}}$-strategies, where $\mathcal{M}_{\text{triv}}$ is



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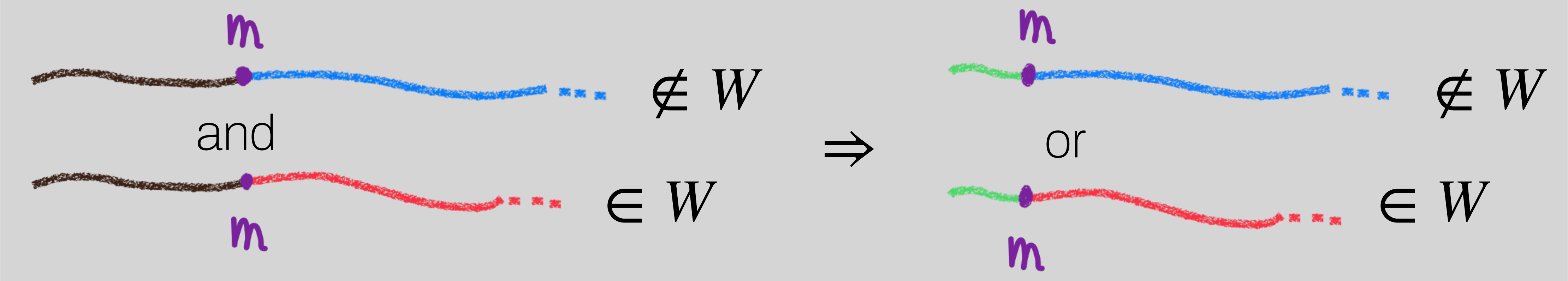

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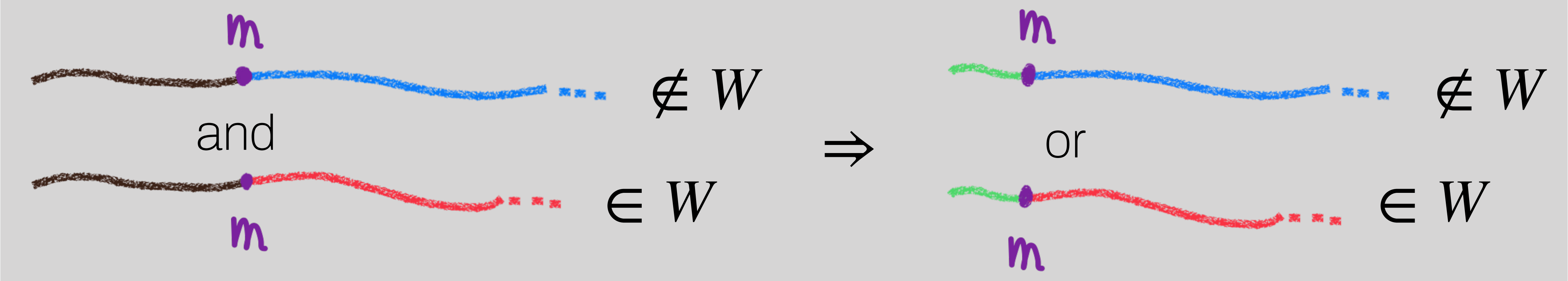
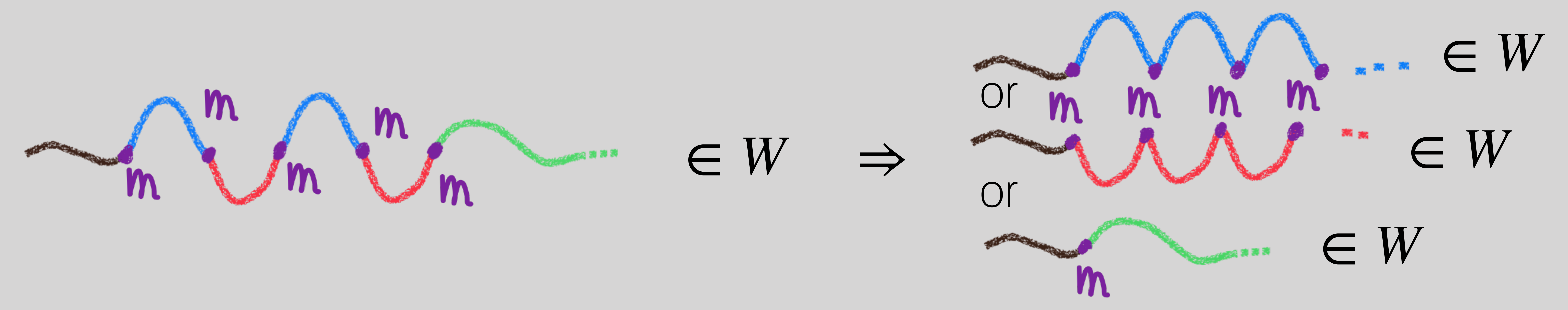


- W is $\mathscr{M}$**-selective** whenever:



Two characterizations

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→ We recover `[GZ05]` with $\mathcal{M} = \mathcal{M}_{\text{triv}}$

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Covered arenas = same properties as product arenas

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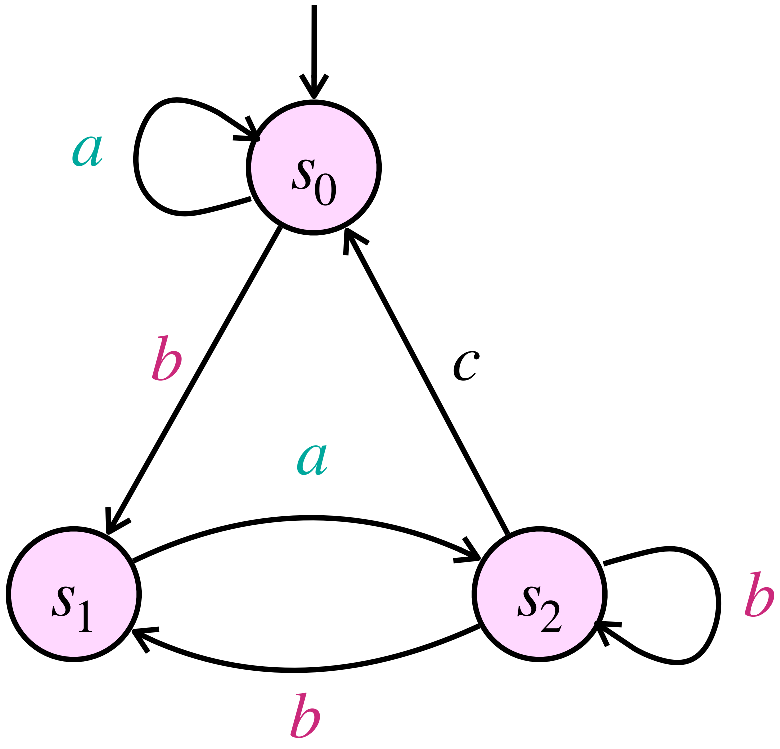

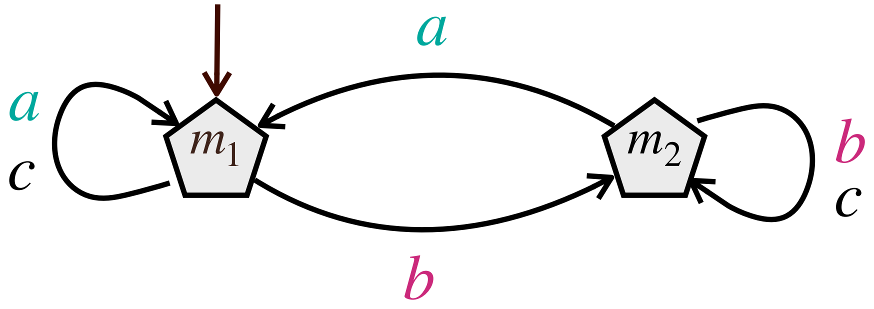

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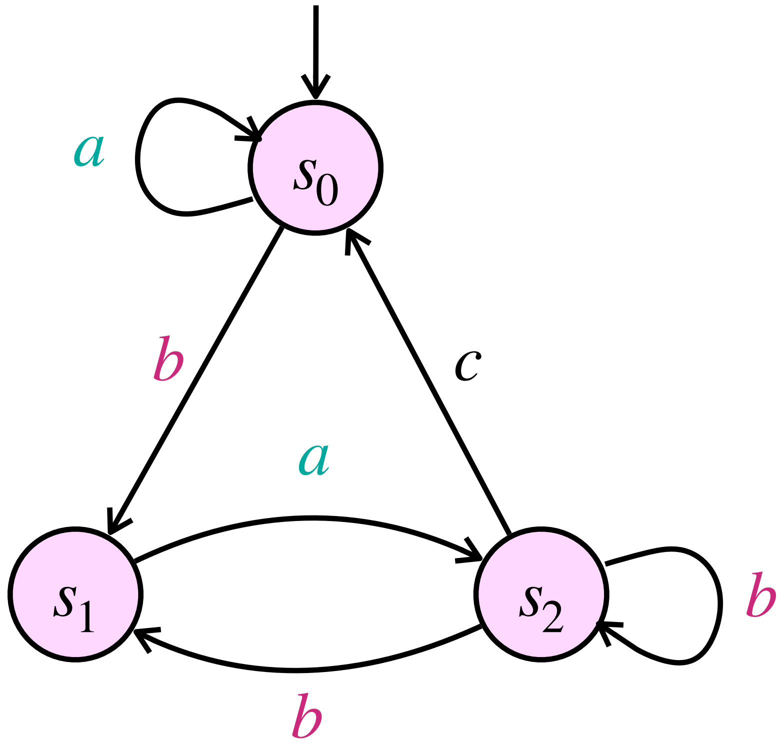

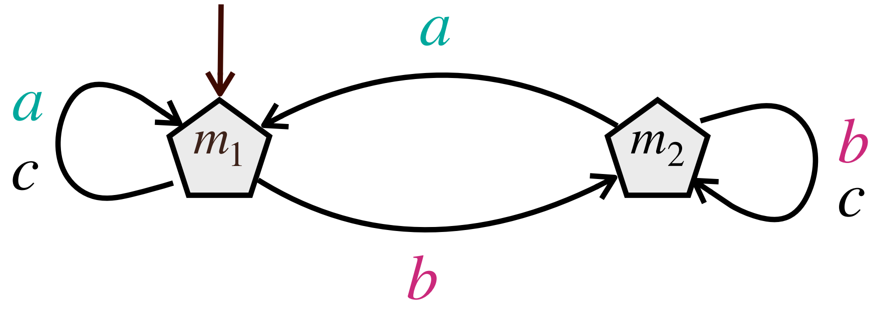

Hence one can apply a `[GZ05]`-like reasoning to $\mathscr{M}$-covered arenas

Applications

Lifting theorem

Strategies based on $\mathscr{M}_i$ suffice for W for P_i in finite P_i-arenas

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W is $(\mathscr{M}_1 \otimes \mathscr{M}_2)$-determined in finite arenas

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  - Conjunction of ω-regular objectives
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$W = \mathrm{Reach}(a) \wedge \mathrm{Reach}(b)$

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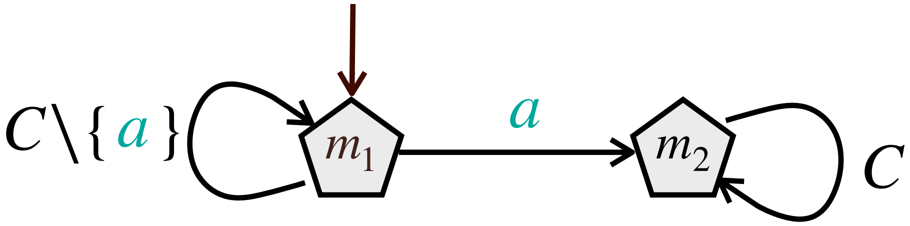
$\mathscr{M}$



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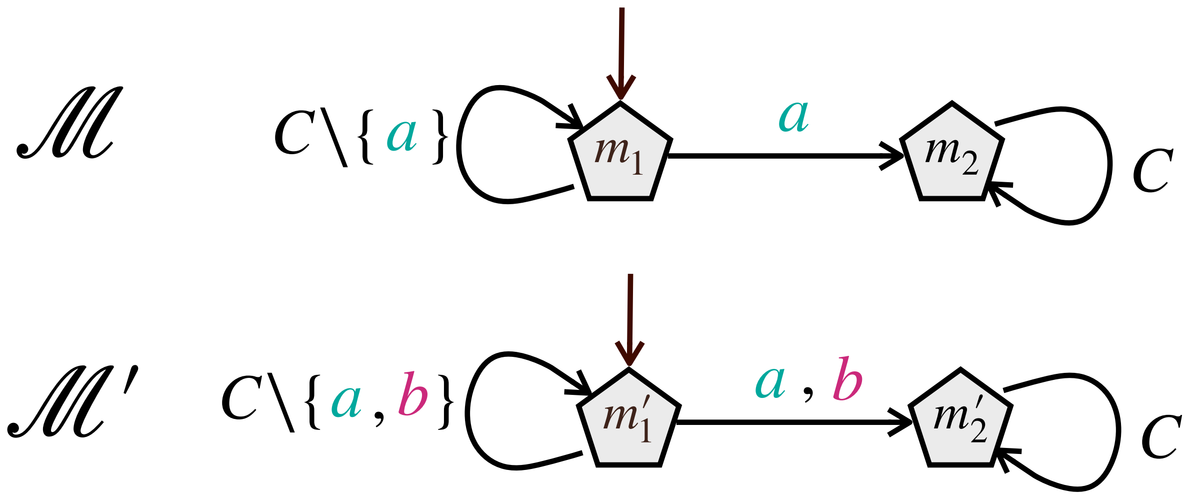

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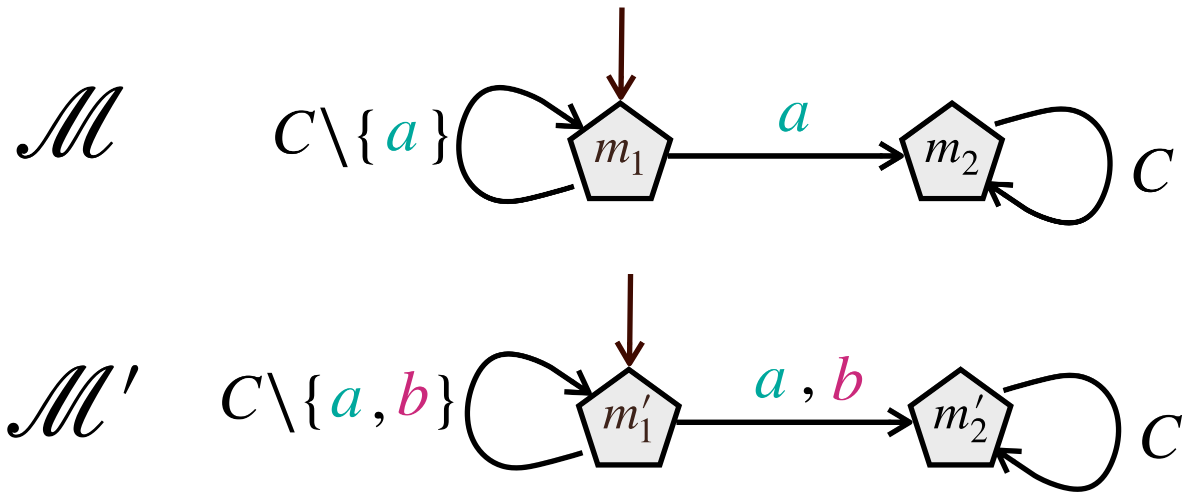

- W is $\mathscr{M}$-monotone and $\mathscr{M}'$-selective
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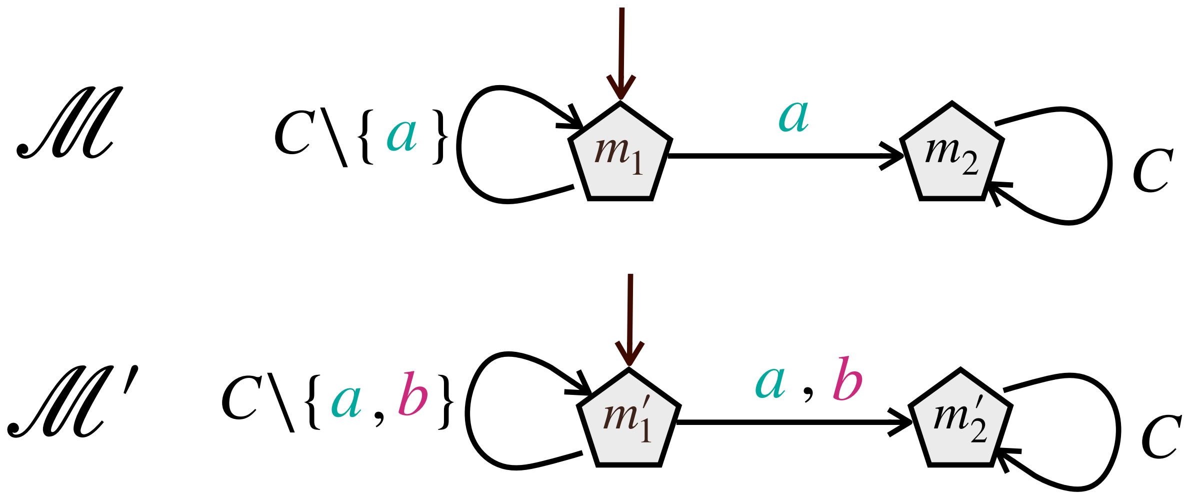

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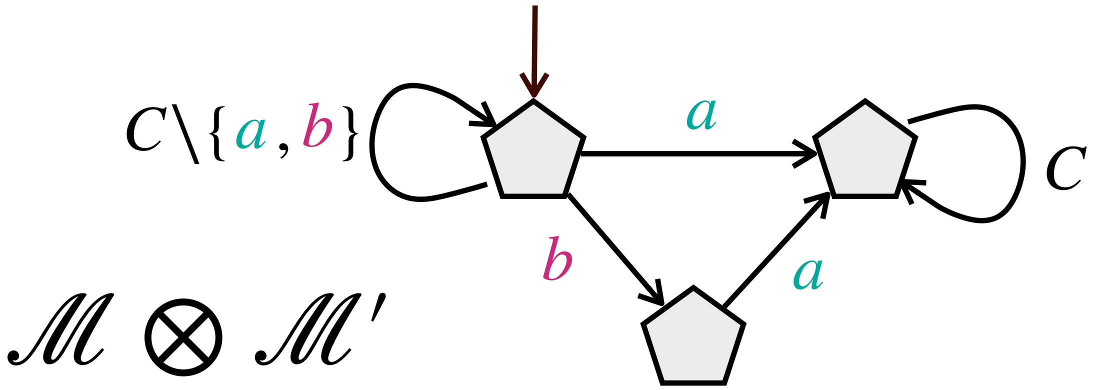

→ Memory $\mathscr{M} \otimes \mathscr{M}'$ is sufficient for both players in all finite games

Partial conclusion

Finite games

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Finite games

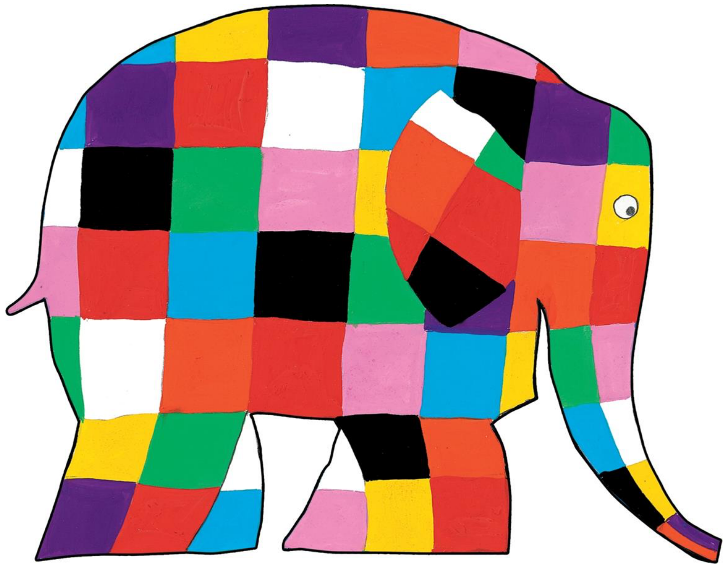
- Complete characterization of winning objectives (and even preference relations) that ensure (chromatic) finite-memory determinacy (for both players)

Partial conclusion

Finite games

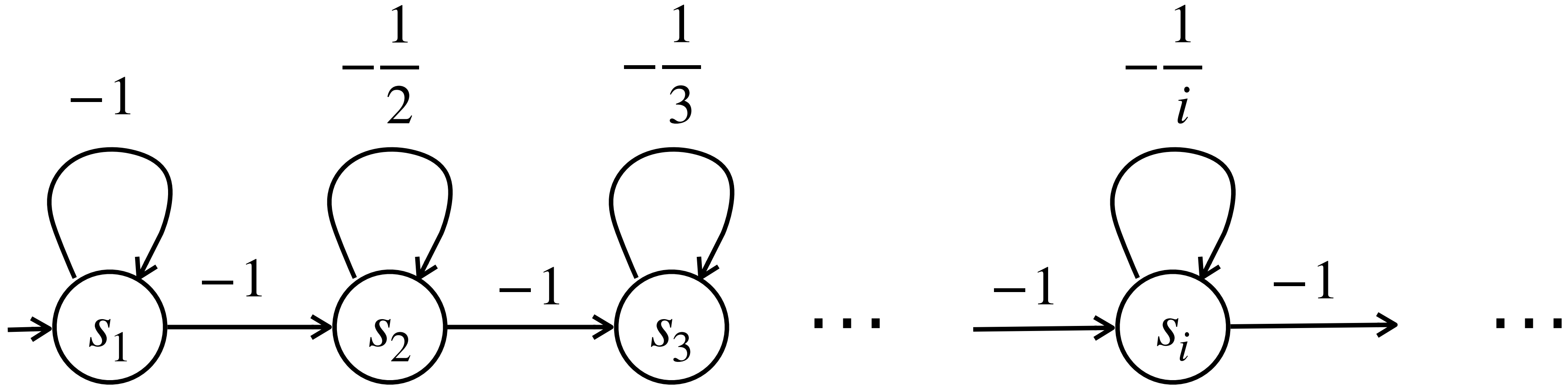
- Complete characterization of winning objectives (and even preference relations) that ensure (chromatic) finite-memory determinacy (for both players)
- One-to-two-player lifts
  (requires chromatic finite memory determinacy in one-player games for both players;
  ensures chromatic finite memory determinacy in two-players games for both players)

Characterizing positional and chromatic finite-memory determinacy in infinite games



The case of mean-payoff

- Objective for P_1: get non-negative (limsup) mean-payoff
- In finite games: **memoryless** strategies are sufficient to win
- In infinite games: **infinite memory** is required to win



A first insight [CN06]

- Let W be a prefix-independent objective.

[CN06] Colcombet and Niwiński. On the positional determinacy of edge-labeled games (ICALP'06).
[Zie98] Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees (TCS).

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1. Positional optimal strategies are sufficient for W in all (infinite) games for both players;
2. W is a parity condition

   That is, there are $n \in \mathbb{N}$ and $\gamma : C \to \{0,1,\ldots,n\}$ such that
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A first insight [CN06]

Limitations

- Let W be a prefix-independent objective.

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Some language theory (1)

- Let $L \subseteq C*$ be a language of finite words

Right congruence

- Given $x, y \in C*$,

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Myhill-Nerode Theorem

- L is regular if and only if $\sim_L$ has finite index;
  - There is an automaton whose states are classes of $\sim_L$, which recognizes L.

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Link with ω-regularity?

- If W is ω-regular, then $\sim_W$ has finite index;
  - The automaton $\mathscr{M}_W$ based on $\sim_W$ is a **prefix-classifier**;
- The converse does not hold (e.g. all prefix-independent languages are such that $\sim_W$ has only one element).

Characterization [BRV22]

- Let $W \subseteq C^\omega$ be an objective.

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- The proof of $\Leftarrow$ is given by [EJ91,Zie98]

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Proof idea for $\Rightarrow$

Assume W is $\mathcal{M}$-determined. Then:

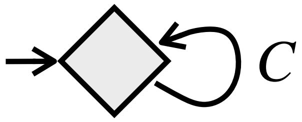
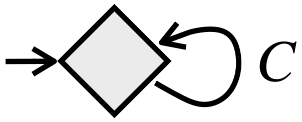
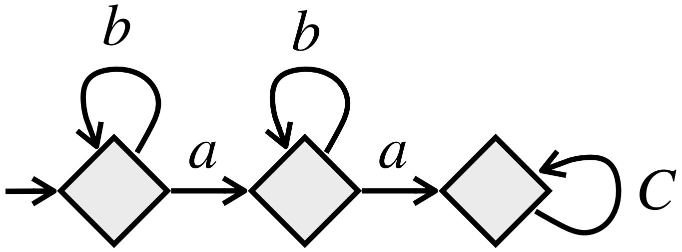
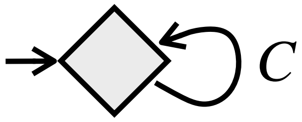
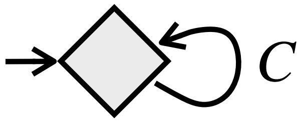
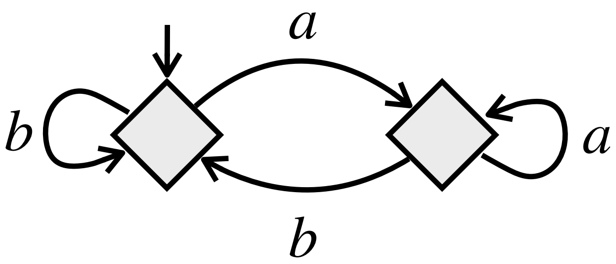
- $\mathcal{M}_W$ is finite (which implies that W is $\mathcal{M}_W$-prefix-independent);
- W is $\mathcal{M}$-cycle-consistent: after a finite word u, if $(w_i)_i$ are winning cycles of $\mathcal{M}$ (after u), then $uw_1w_2w_3\cdots$ is winning; Idem for losing cycles

→ W is $(\mathcal{M} \otimes \mathcal{M}_W)$-prefix-independent and $(\mathcal{M} \otimes \mathcal{M}_W)$-cycle-consistent

→ Hence W can be recognized by a DPA built on top of $\mathcal{M} \otimes \mathcal{M}_W$
(relies on ordering cycles according to how good they are for winning)

Difficult part of the proof

Examples

Objective W	Prefix classifier $\mathscr{M}_W$	Memory $\mathscr{M}$
Parity objective		
$C = \{a, b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a, b\}$ $W = C^*(ab)^\omega$		

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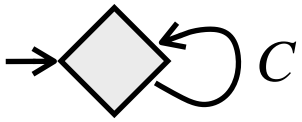
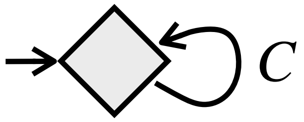
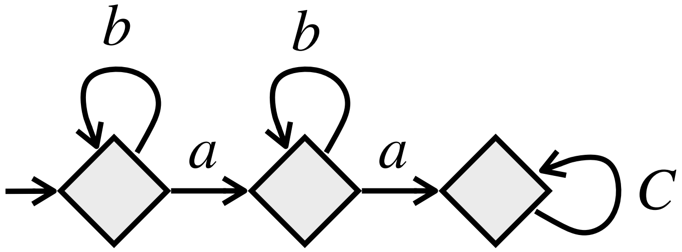
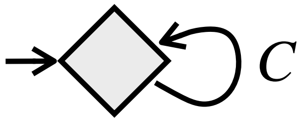
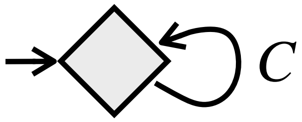
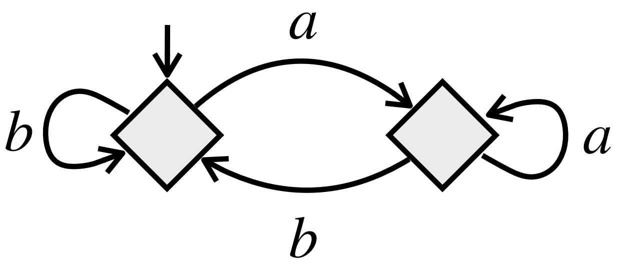
Parity objective		 $\mapsto \{0,1,\ldots,n\}$
$C = \{a,b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a,b\}$ $W = C^*(ab)^\omega$		

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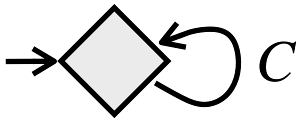
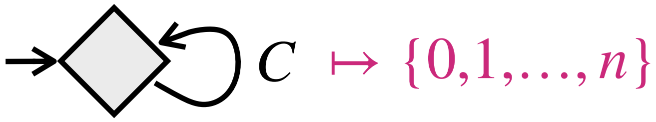
<tr><td>Parity objective</td><td>
</td><td>
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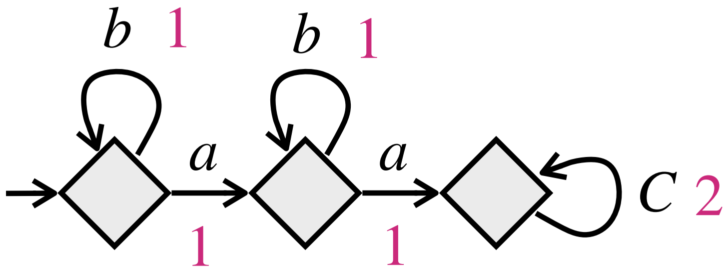
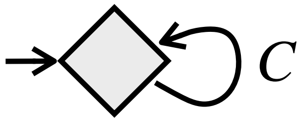
$W = b^*ab^*aC^\omega$</td><td>
</td><td>
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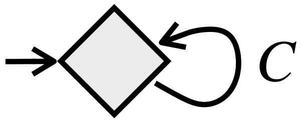
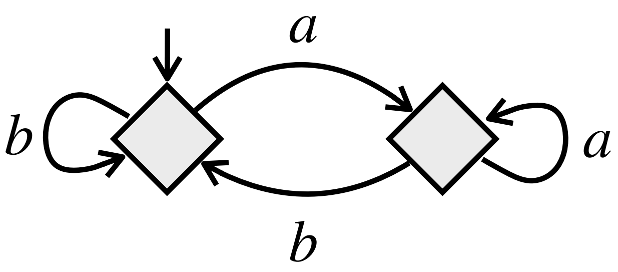
$W = C^*(ab)^\omega$</td><td>
</td><td>
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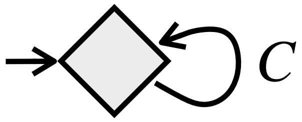
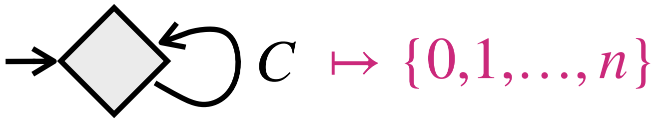
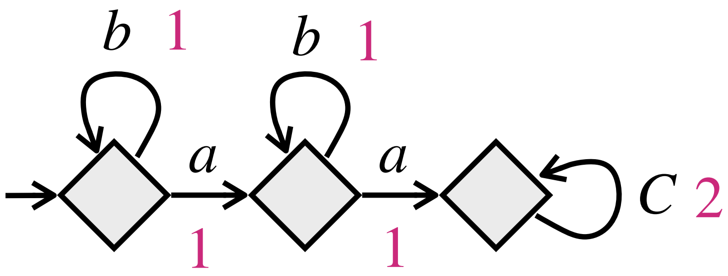
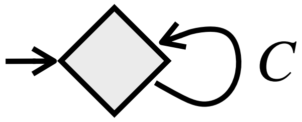
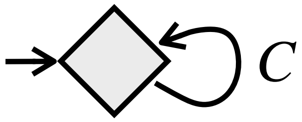
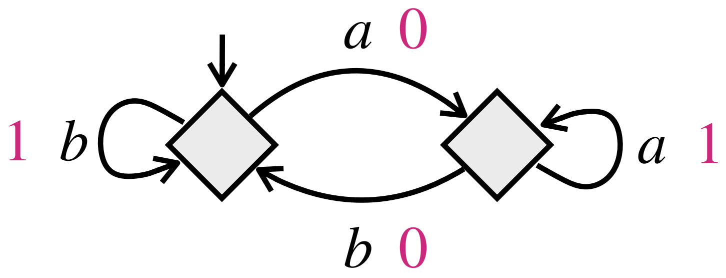
Parity objective		
$C = \{a, b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a, b\}$ $W = C^*(ab)^\omega$		

Corollary

Lifting theorem

If W and W^c are finite-memory-determined in one-player infinite games, then W and W^c are finite-memory-determined in two-player infinite games.

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Very powerful and extremely useful in practice

- Easier to analyse the one-player case (graph reasoning)
- Lift to two-player games via the theorem

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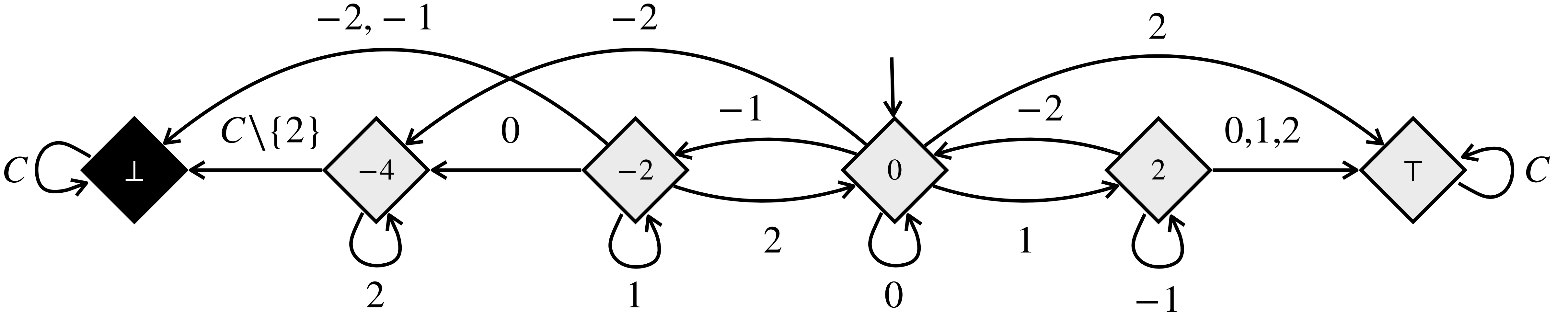

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Infinite games

- Complete characterization of winning objectives that ensure chromatic finite-memory determinacy in infinite games = ω-regular
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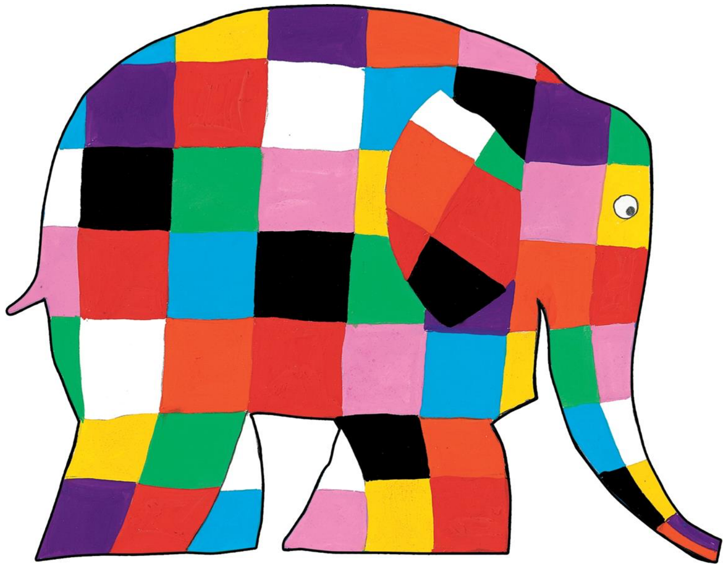
- Further questions:
  - Different results when assuming finite branching?

Going further?



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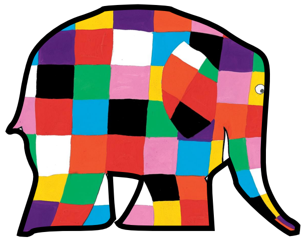

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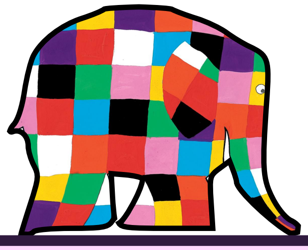

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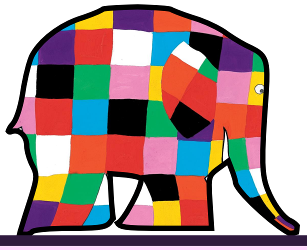

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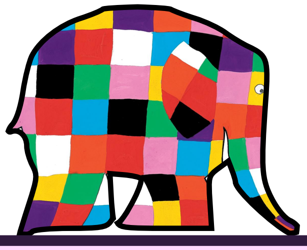

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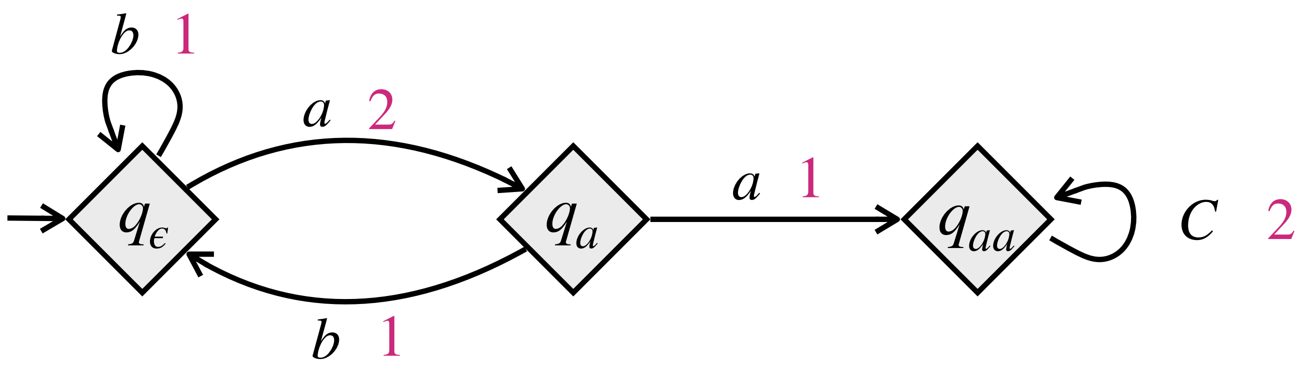
→ Precise memory of the two players for ω-regular objectives?
(we will see it is non-trivial in general)

Example

$W = (b^*a)^\omega \cup C^*aaC^\omega$



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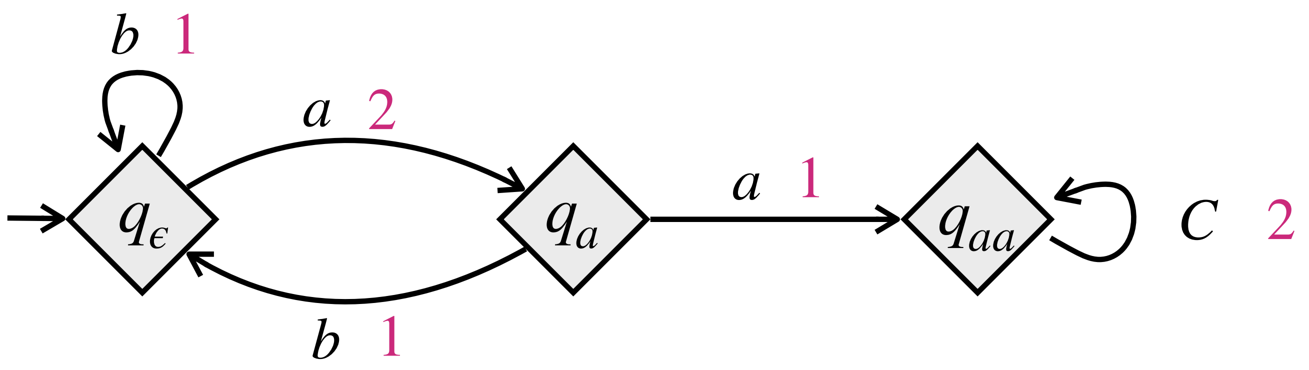

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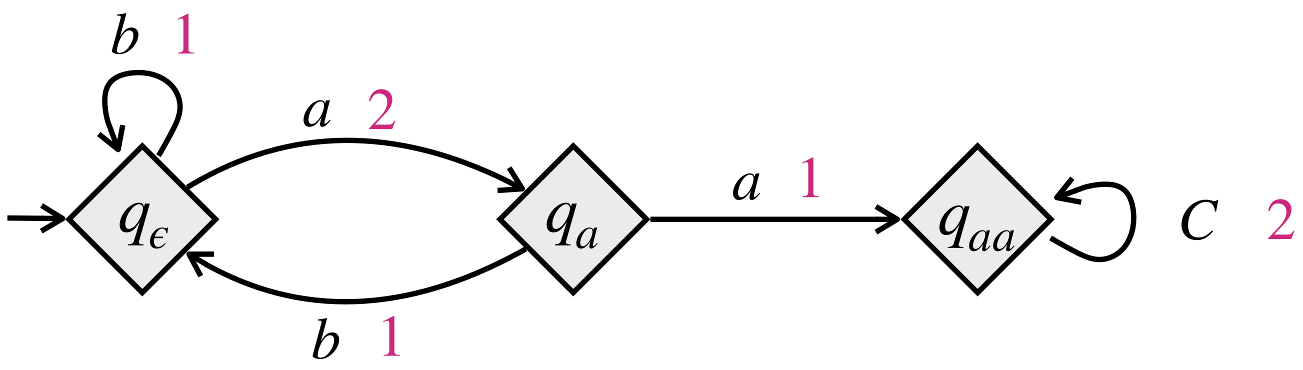

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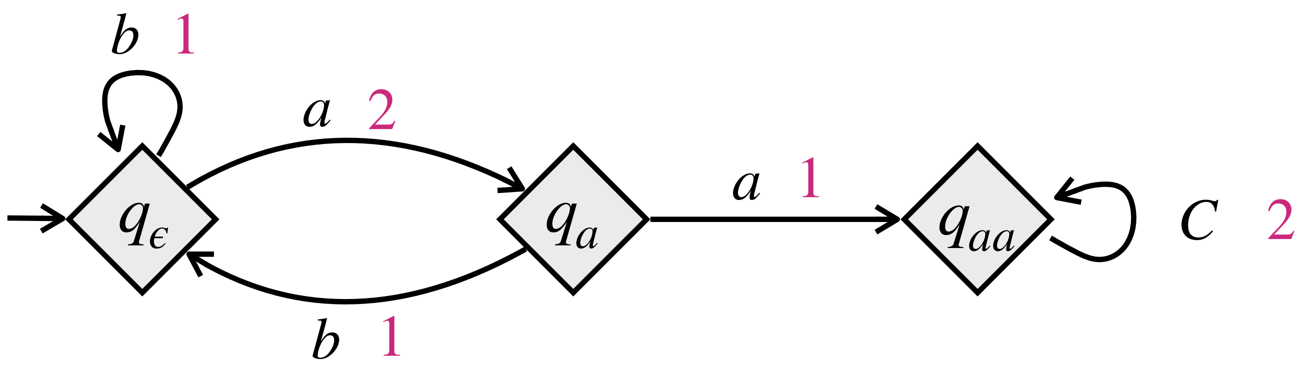

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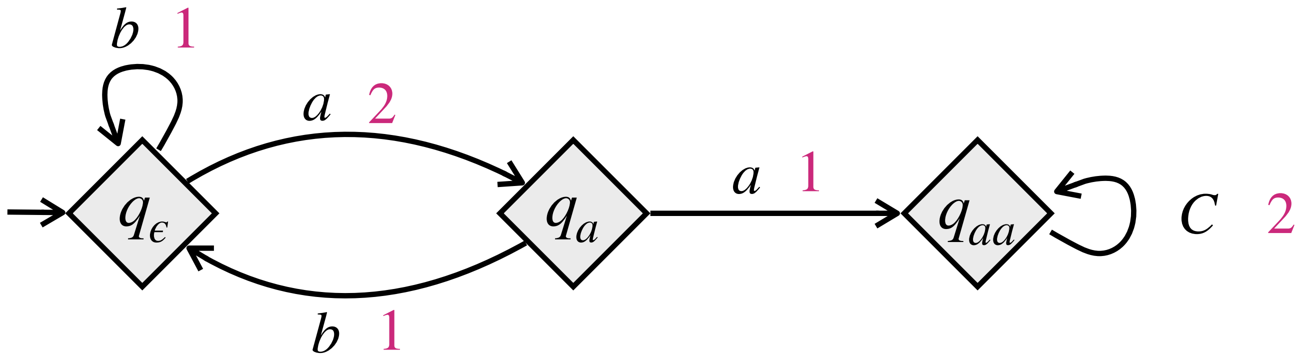

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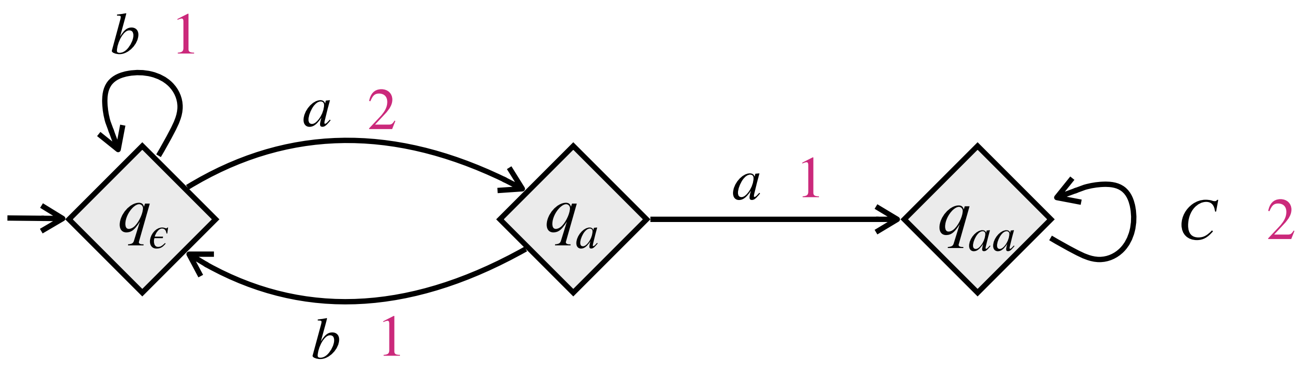

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  - P_2 requires just two states of memory: q_ϵ and q_a

The special case of objectives given by DBA [BCRV22]

[BCRV22] Bouyer, Casares, Randour, Vandenhove. Half-positional objectives recognized by deterministic Büchi automata (CONCUR'22)

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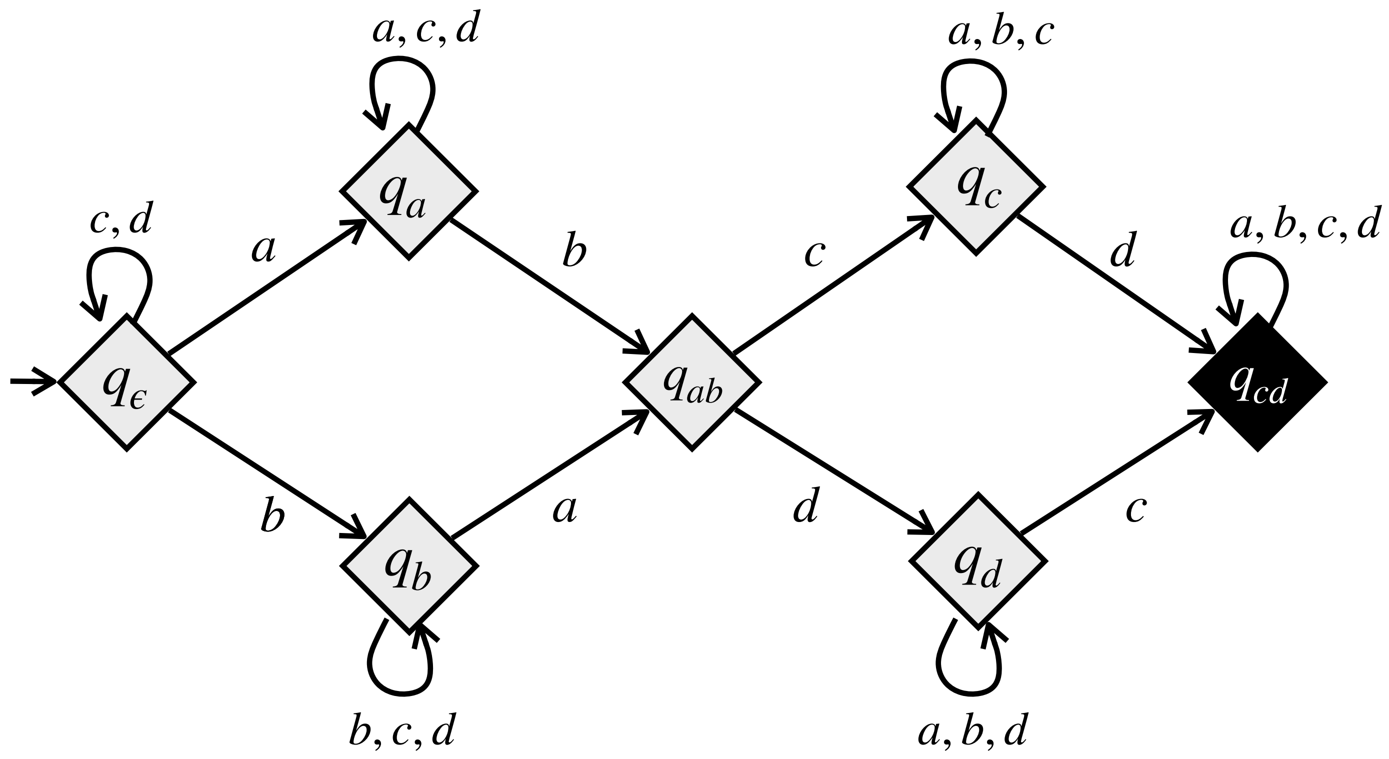
Half-positionality of W can be decided in PTIME

An objective W defined by a DBA is half-positional if and only if:

1. W is monotone;
2. W is progress consistent: if w_2 is a progress after w_1, then $w_1 w_2^\omega$ is winning;
3. W is recognized by a DBA built on top of its prefix classifier

[BCRV22] Bouyer, Casares, Randour, Vandenhove. Half-positional objectives recognized by deterministic Büchi automata (CONCUR'22)

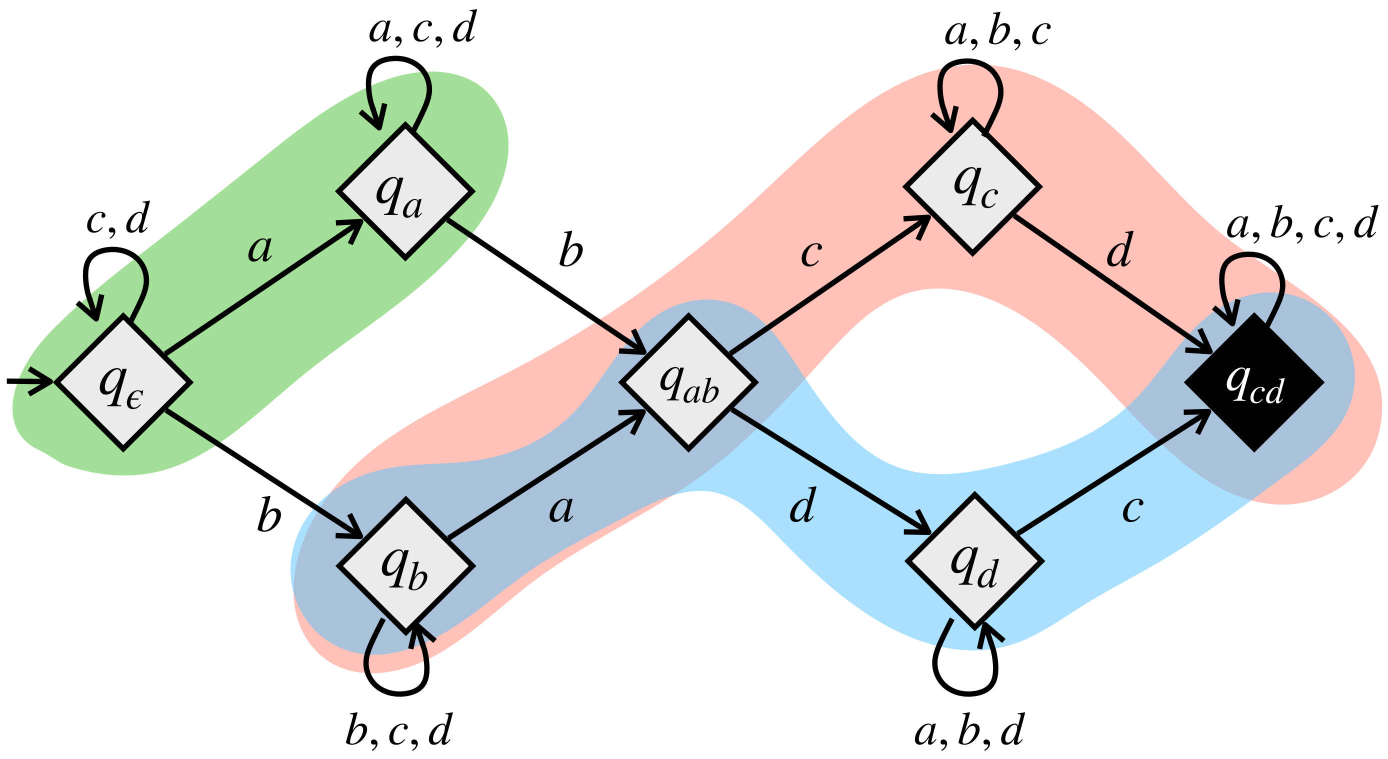
Regular safety and reachability objectives [BFRV22]



$W =$ avoid the rightmost state

[BFRV22] Bouyer, Fijalkow, Randour, Vandenhove. How to Play Optimally for Regular Objectives? (Submitted)

Regular safety and reachability objectives [BFRV22]



$W =$ avoid the rightmost state

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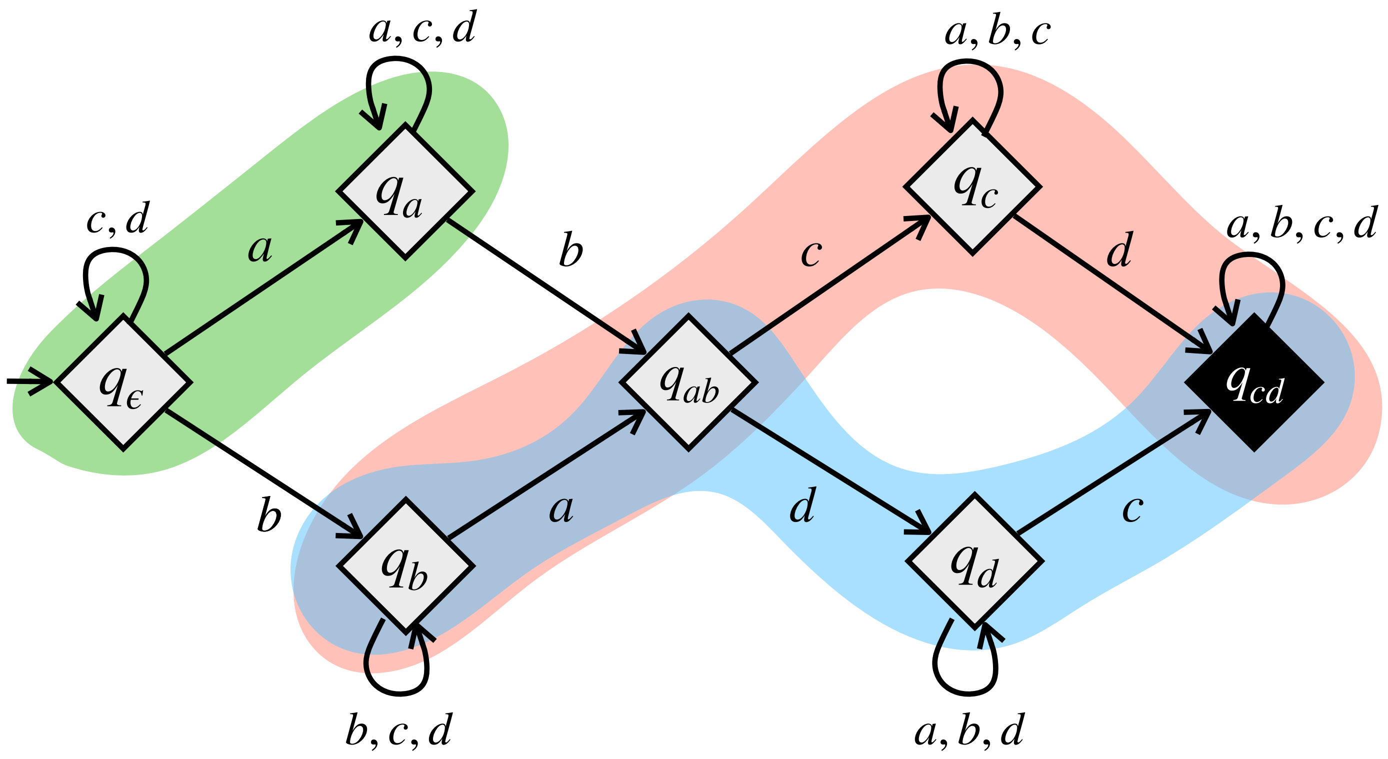

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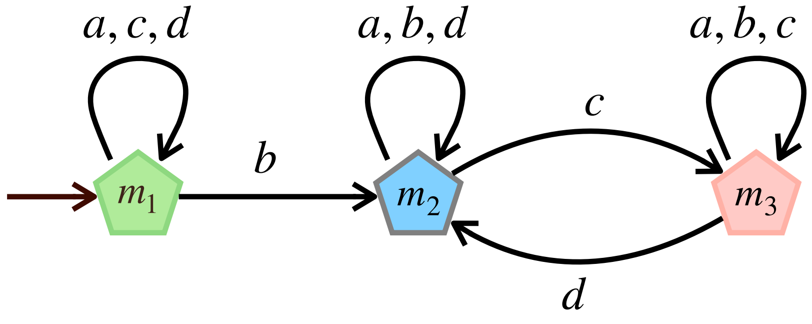

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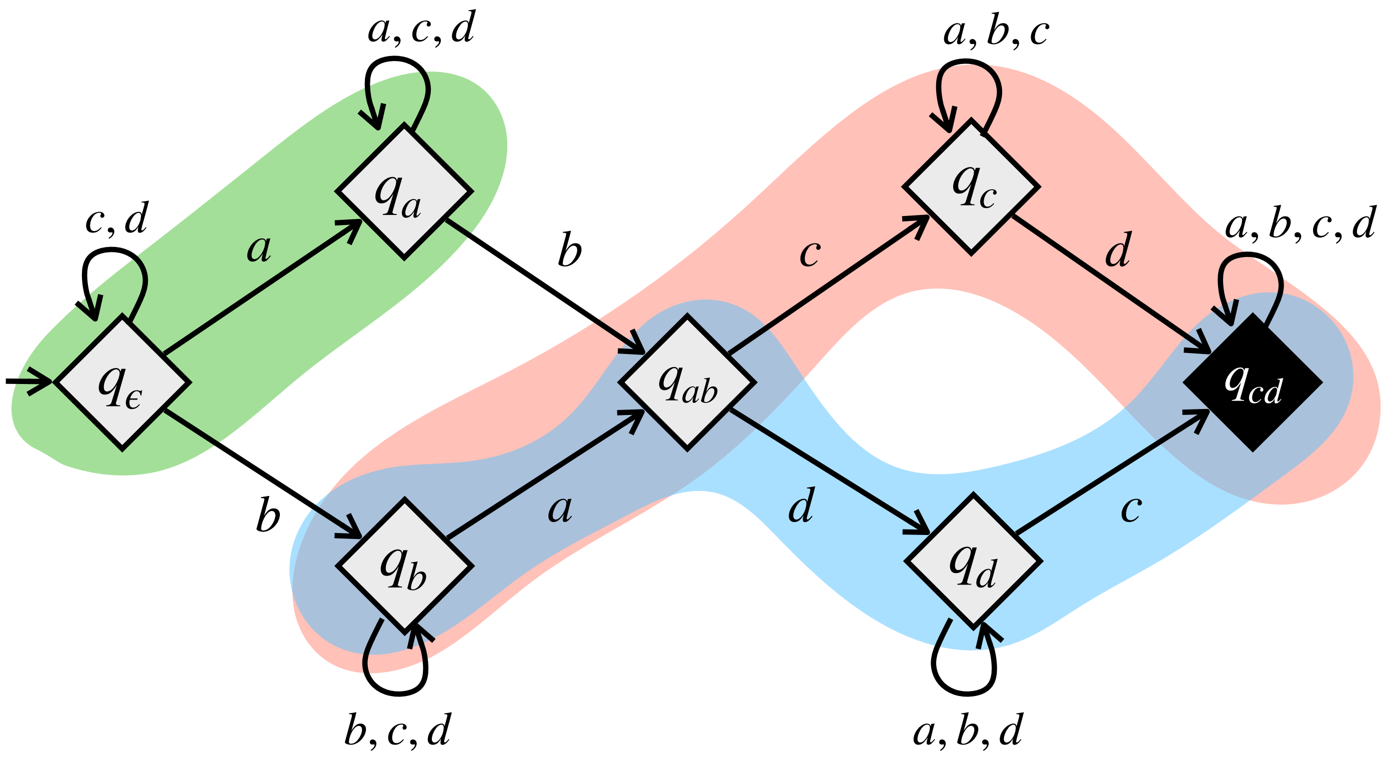

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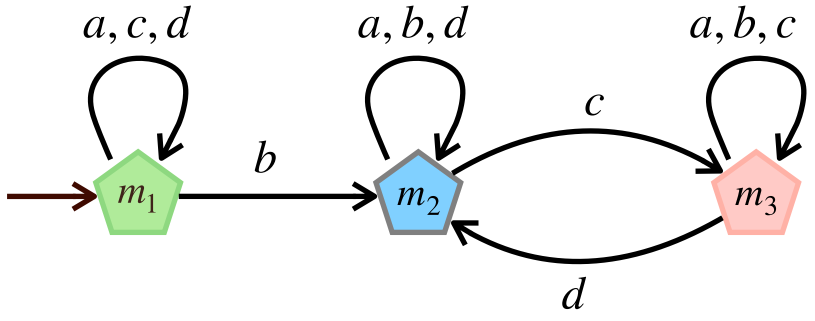

Tightest memory to win W

It is NP-complete to decide whether there is a memory structure of size k that is sufficient to win a regular safety/reachability objective.

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Double lift

- Let $W \subseteq C^\omega$ be a regular reachability or safety objective

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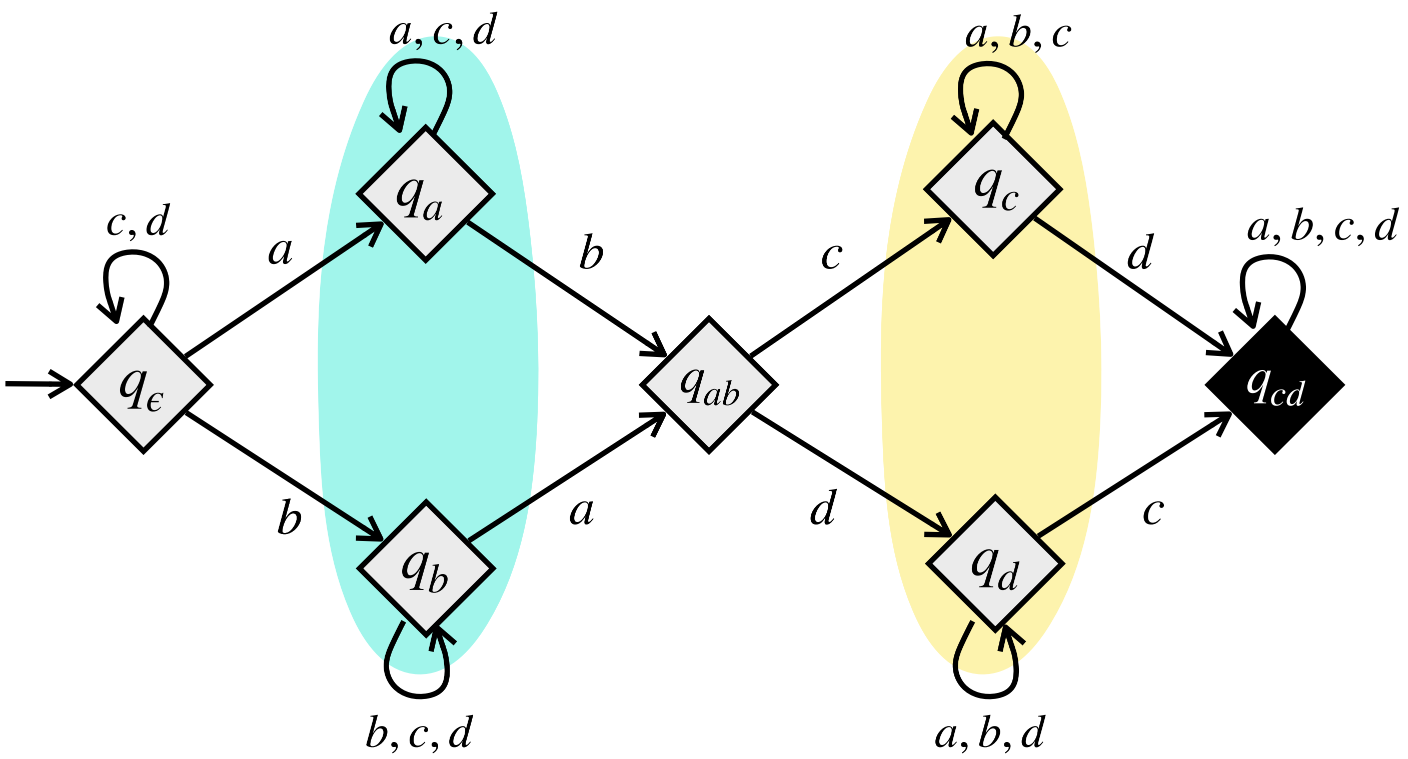
Very powerful and extremely useful in practice

- Easy to analyse the one-player finite case (finite graph reasoning)
- Lift to infinite two-player games via the theorem

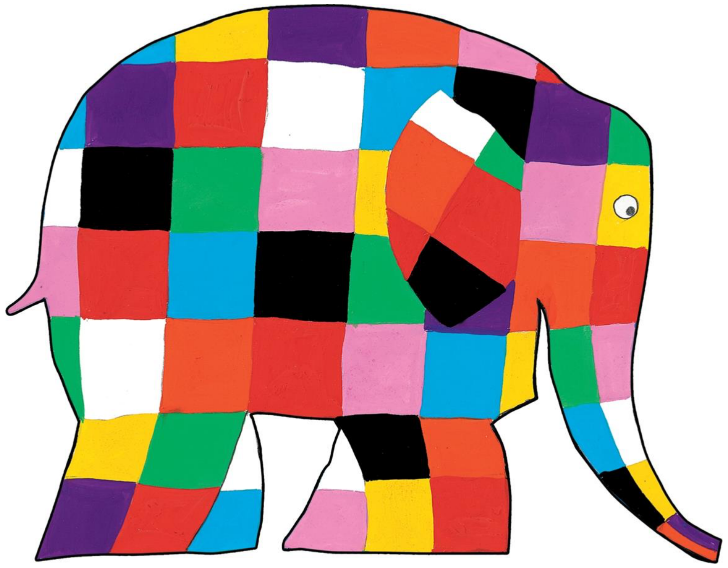
What about chaotic memory?

- Chaotic memory is more difficult to grasp
- In the previous example, only two memory states are sufficient (size of the largest antichain) [CFH14]



[CFH14] Colcombet, Fijalkow, Horn. Playing safe (FSTTCS'14)

Conclusion



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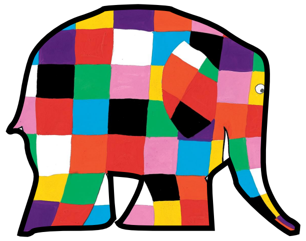

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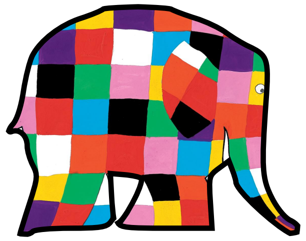

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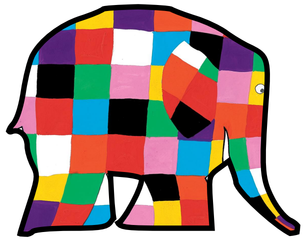

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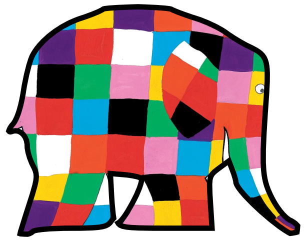

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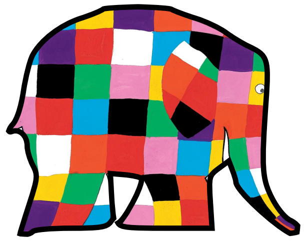

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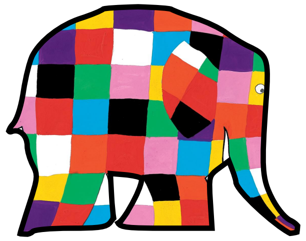

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Quite active area of research

[CCL22] Casares, Colcombet, Lehtinen.On the size of good-for-game Rabin automata and its link with the memory in Muller games (ICALP’22)

[Ohl22] Ohlmann. Characterizing positionality in games of infinite duration over infinite graphs (LICS’22)