Laboratoire Méthodes Formelles

On the Probabilistic and Statistical Verification of Infinite Markov Chains

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CNRS, ENS Paris-Saclay
France

Joint work with Benoît Barbot (LACL) and Serge Haddad (LMF)

Work partly supported by ANR projects MAVeriQ and BisoUS

General purpose

Design algorithms to estimate probabilities in some **infinite-state** Markov chains, **with guarantees**

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Our contributions

- Review two existing approaches (approximation algorithm and estimation algorithm) and specify the required hypothesis for correctness
- Propose an approach based on **importance sampling** and **abstraction** to partly relax the hypothesis
- Analyze empirically the approaches

Discrete-time Markov chains

Discrete-time Markov chain (DTMC)

$\mathscr{C} = (S, s_0, \delta)$ with S at most denumerable, $s_0 \in S$ and $\delta : S \to \mathrm{Dist}(S)$

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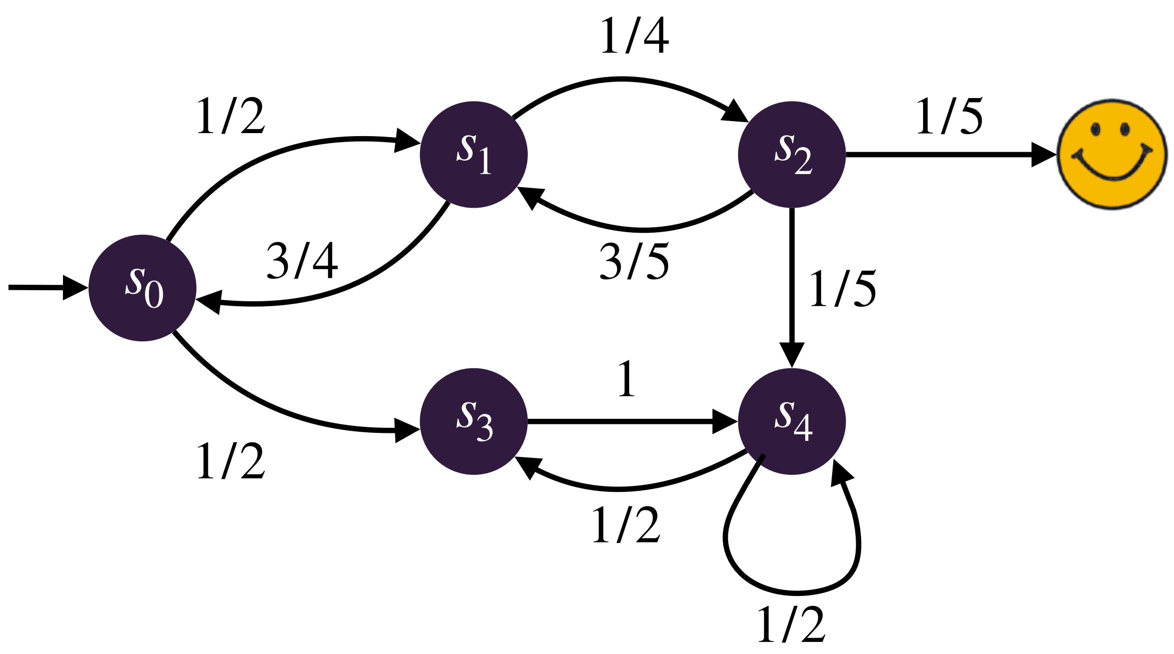

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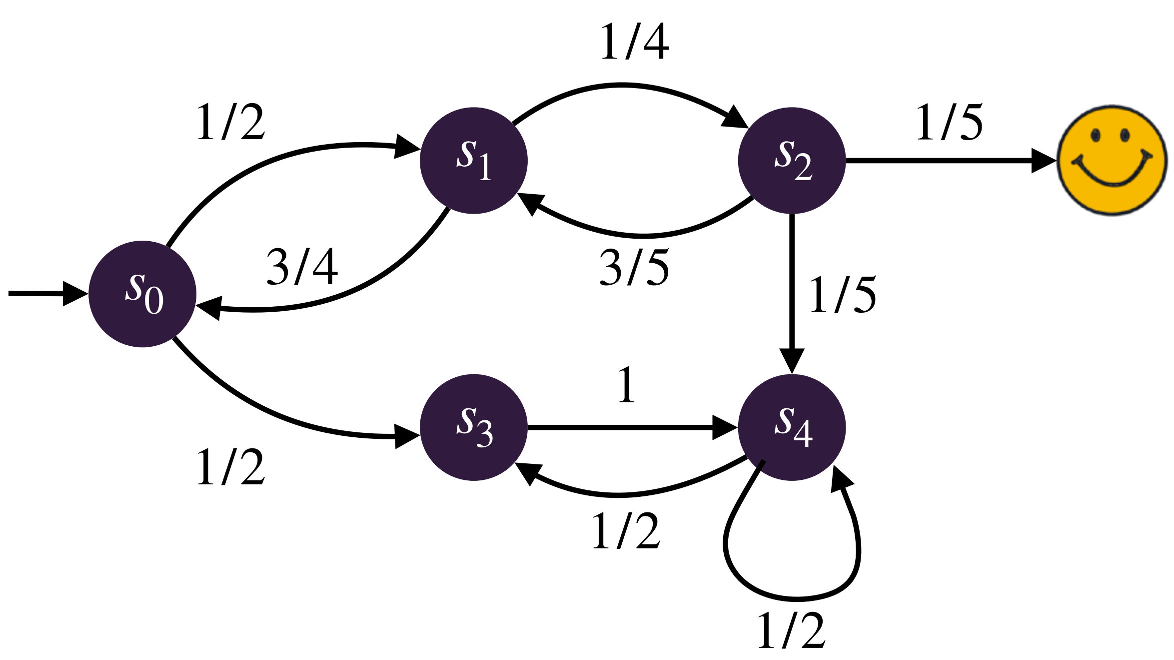

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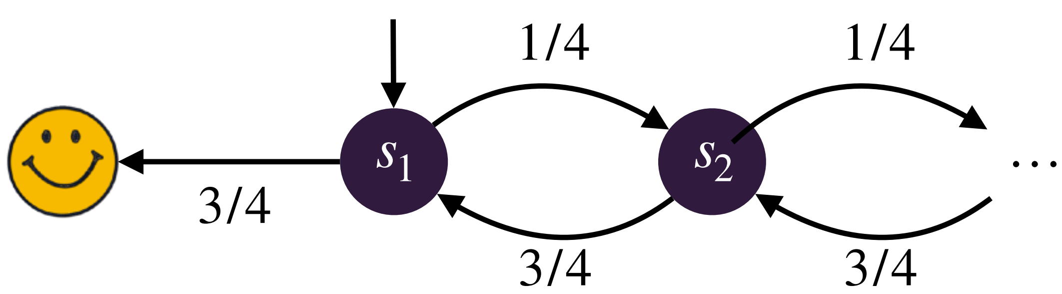

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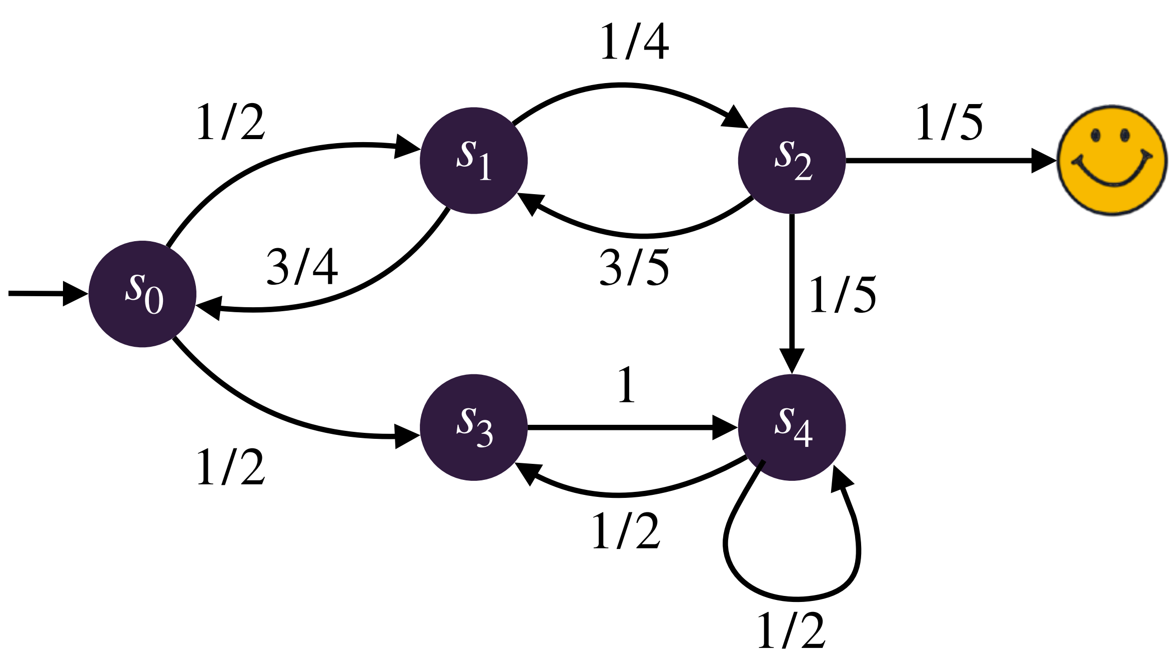

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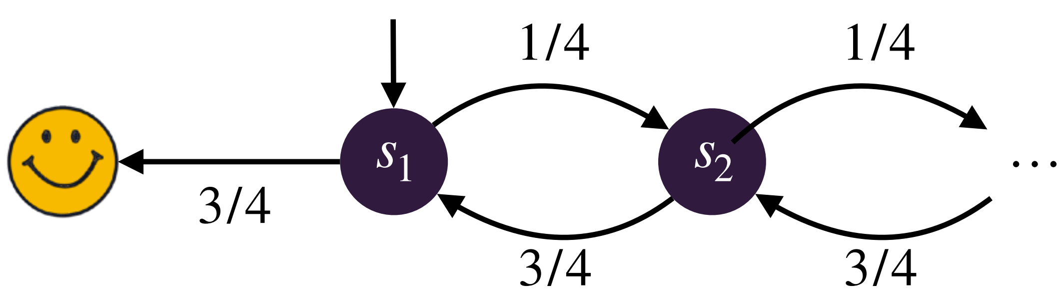

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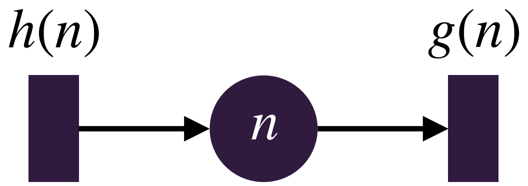
High-level models for (infinite) Markov chains

- Queues



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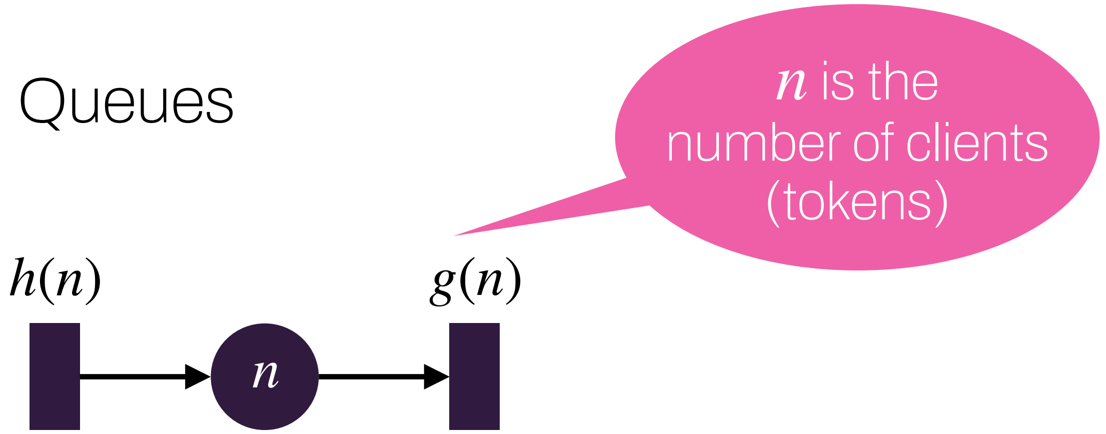

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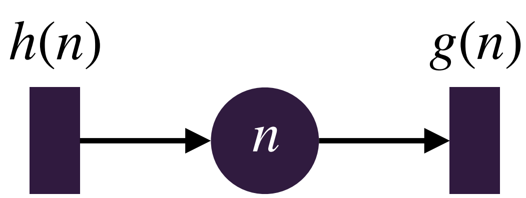

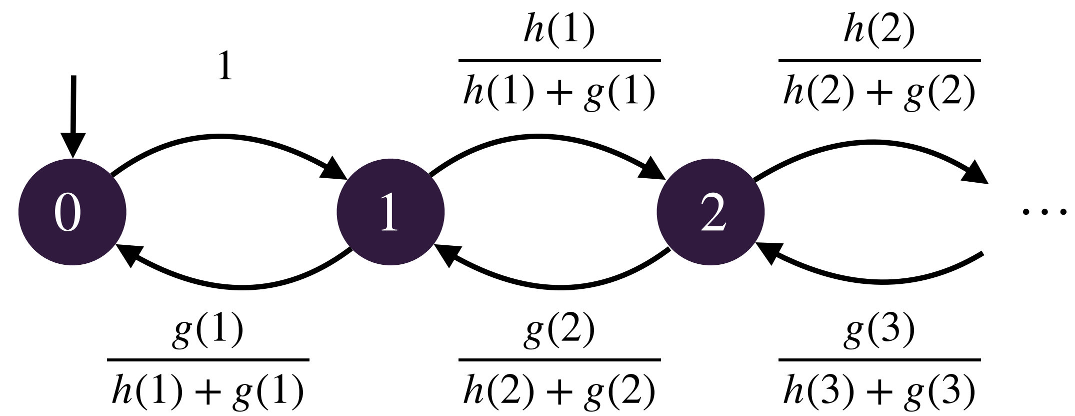

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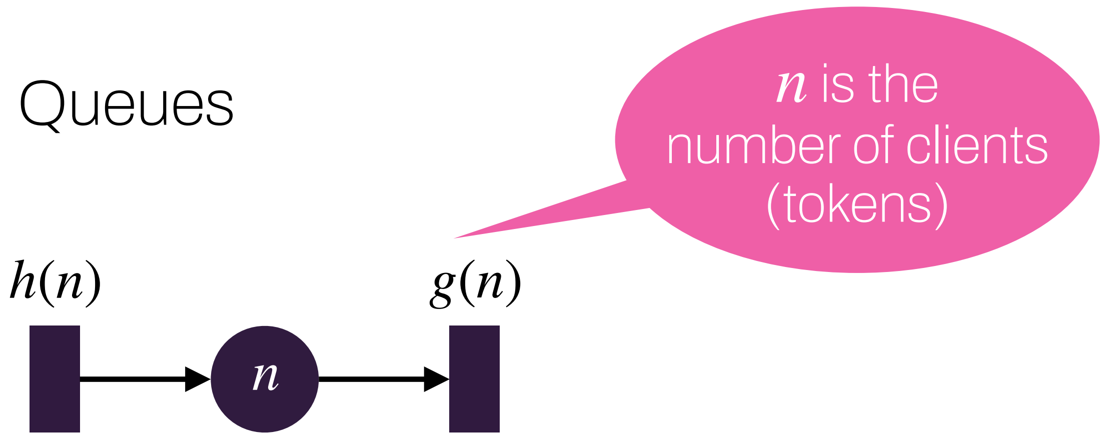

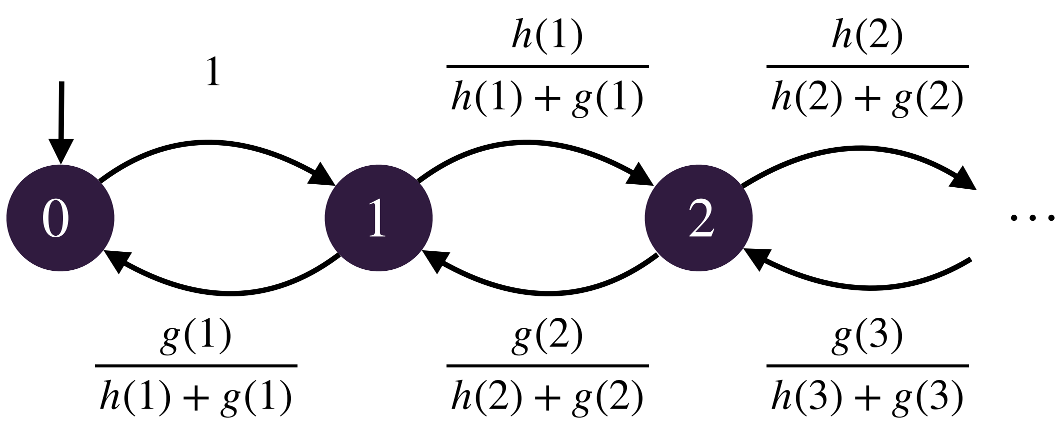

- Probabilistic pushdown automata

$$A \xrightarrow{1} C \quad A \xrightarrow{n} BB \quad B \xrightarrow{5} \varepsilon$$

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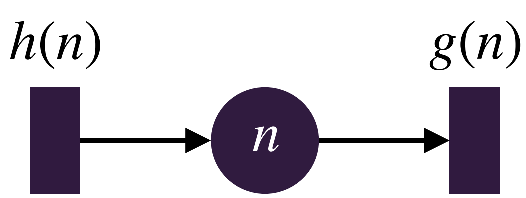

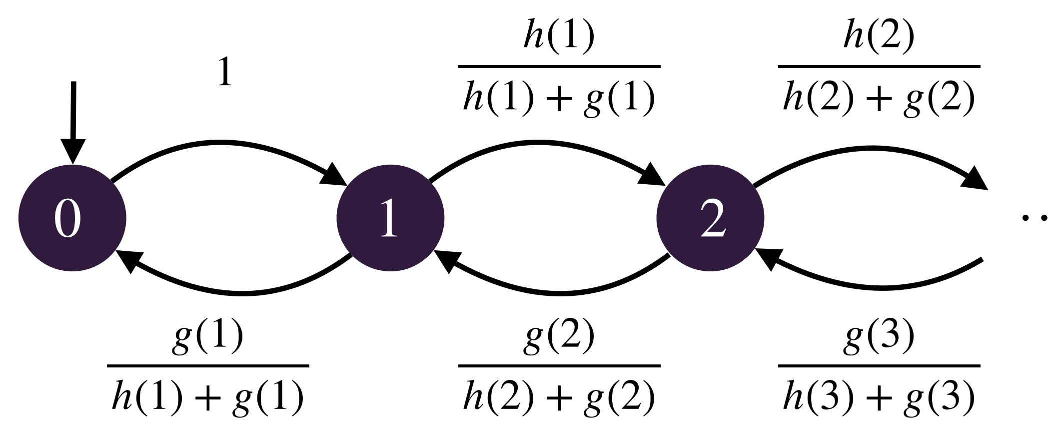

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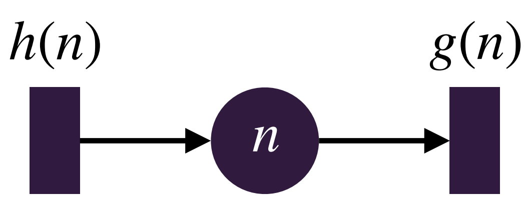

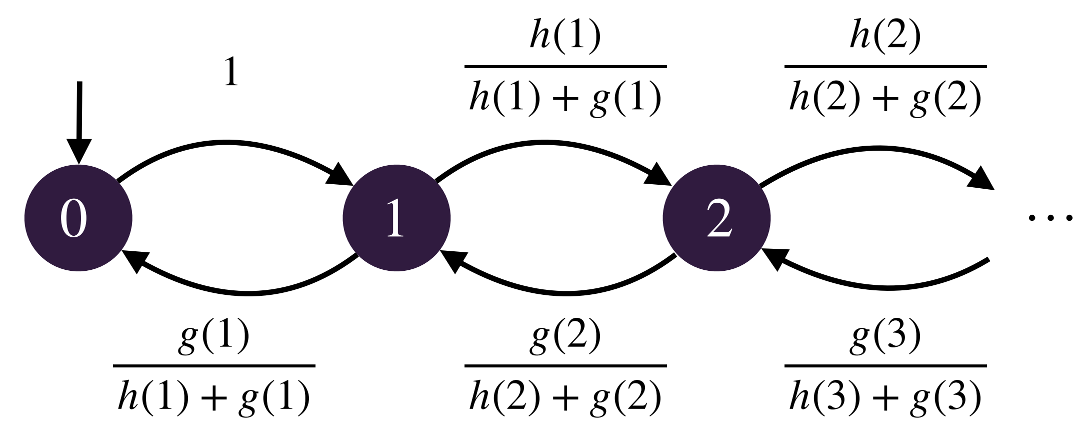

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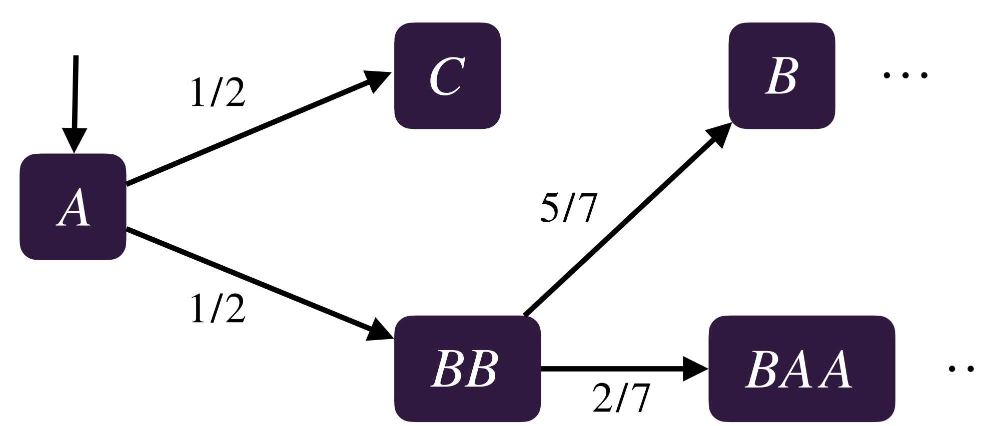
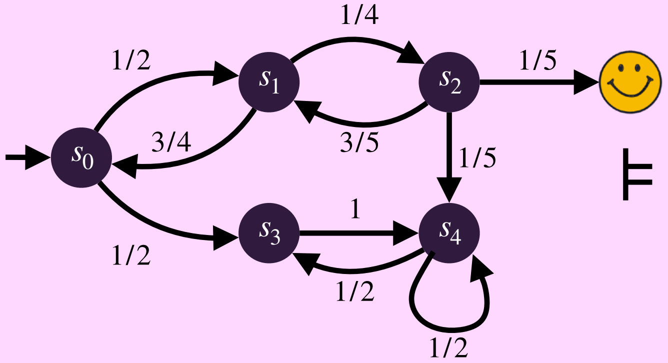


Quantitative analysis of Markov chains

Compute $\mathbb{P}\Big($  $\models \mathbf{F}$ ☺ $\Big)$

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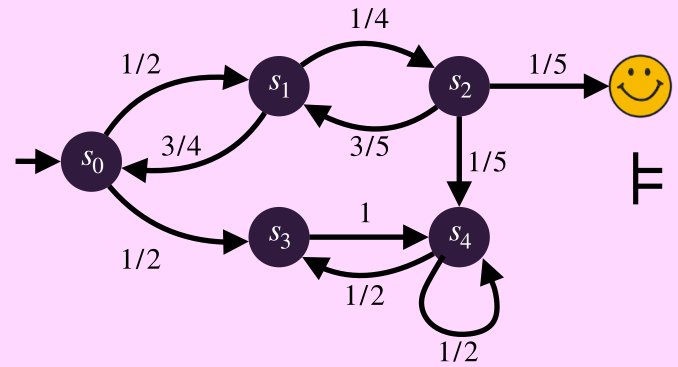
Compute $\mathbb{P}\Big($ 

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Closed-form solution

- Random walk of parameter $p > 1/2$:
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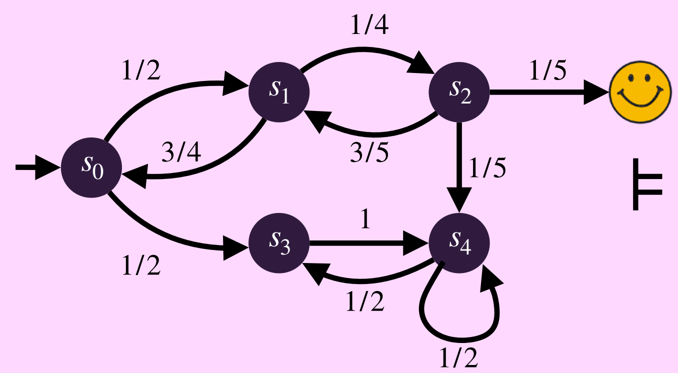
Compute $\mathbb{P}\Big($ 

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Apply a numerical method [RKPN04]

- $$x_s = \begin{cases} 1 & \text{if } s = \text{☺} \\ 0 & \text{if } s \not\models \exists \mathbf{F}\text{☺} \\ \sum_t \mathbb{P}(s \to t) \cdot x_t & \text{otherwise} \end{cases}$$
- $\mathbb{P}_{s_0}(\mathbf{F}$ ☺ $) = 1/19$
- System must be finite
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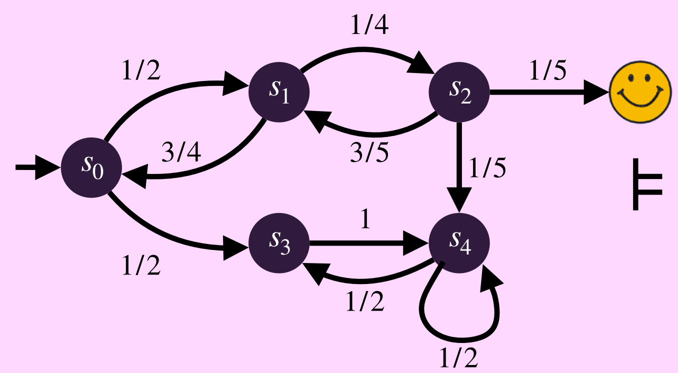
Compute $\mathbb{P}\Big($ 

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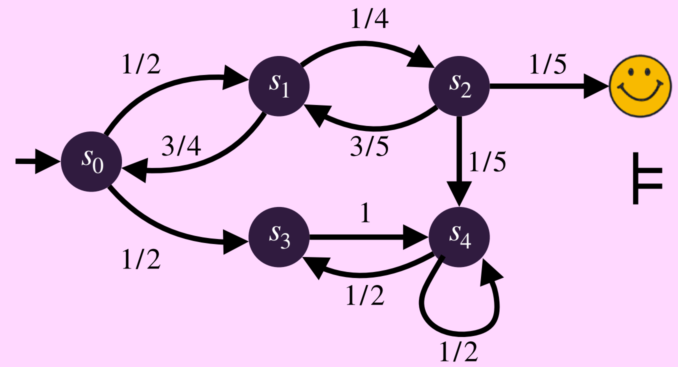
Compute $\mathbb{P}\Big($ 

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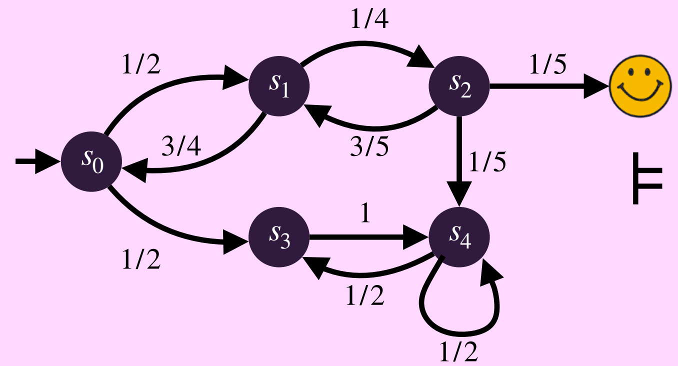
Compute $\mathbb{P}\Big($ 

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- Specific approaches for **decisive** Markov chains

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Decisiveness

☹️ $= \{s \in S \mid s \not\models \exists \mathbf{F}$ 🙂 $\}$

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A DTMC $\mathscr{C}$ is decisive from s w.r.t. 🙂 if $\mathbb{P}_s(\mathbf{F}$ 🙂 $\vee \mathbf{F}$ ☹️ $) = 1$

[ABM07] P.A. Abdulla, N. Ben Henda, R. Mayr. Decisive Markov chains (LMCS, 2007)

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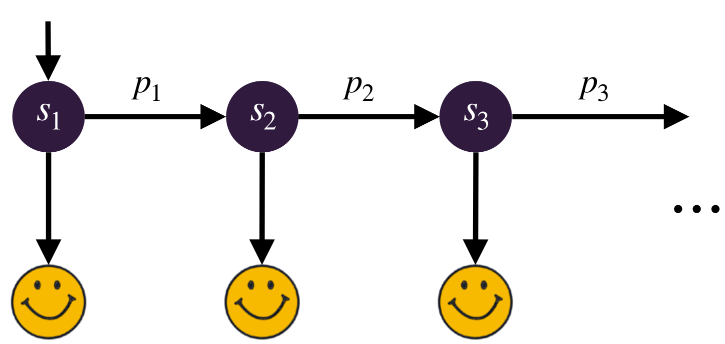

- $\mathbb{P}(\mathbf{G}\neg$ 🙂$) = \prod_{i \geq 1} p_i$
- Decisive iff this product equals 0

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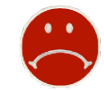
Decisiveness

 $= \{s \in S \mid s \not\models \exists \mathbf{F}$  $\}$

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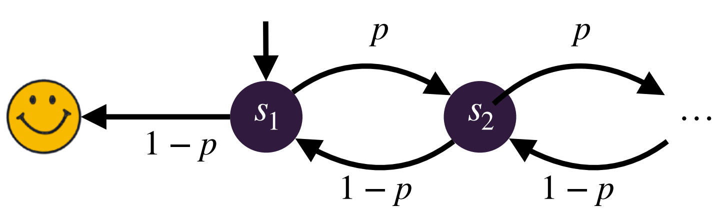

- Recurrent random walk ($p \leq 1/2$): decisive
- Transient random walk ($p > 1/2$): not decisive

[ABM07] P.A. Abdulla, N. Ben Henda, R. Mayr. Decisive Markov chains (LMCS, 2007)

Deciding decisiveness?

Classes where decisiveness can be decided

- Probabilistic pushdown automata with constant weights [ABM07]
- Random walks with polynomial weights [FHY23]
- So-called probabilistic homogeneous one-counter machines with polynomial weights (this extends the model of quasi-birth death processes) [FHY23]

[ABM07] P.A. Abdulla, N. Ben Henda, R. Mayr. Decisive Markov chains (LMCS, 2007)
[FHY23] A. Finkel, S. Haddad, L. Yé. About decisiveness of dynamic probabilistic models (CONCUR'23)

Approximation scheme

- Aim: compute probability of $\mathbf{F}$ 

-  $= \{ s \in S \mid s \not\models \exists \mathbf{F}$  $\}$

[IN97] P. Iyer, M. Narasimha. Probabilistic lossy channel systems (TAPSOFT'97)
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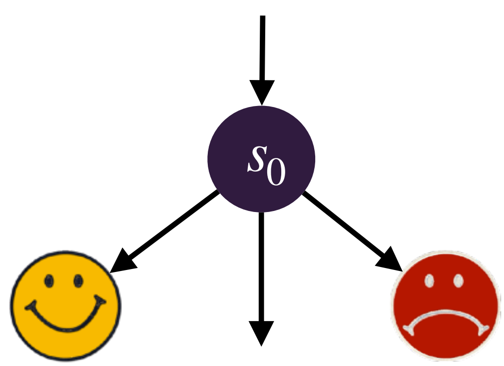

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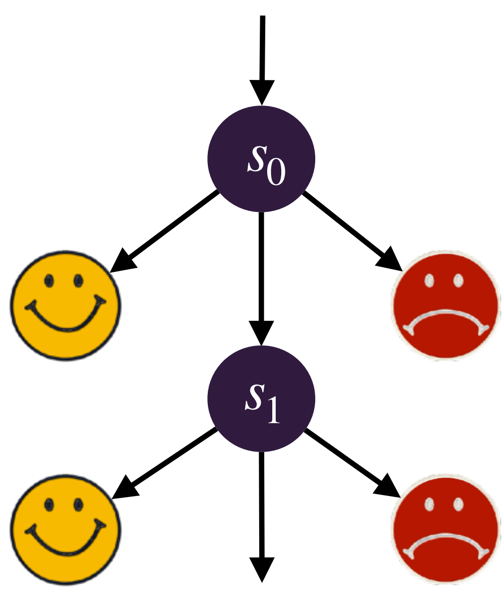

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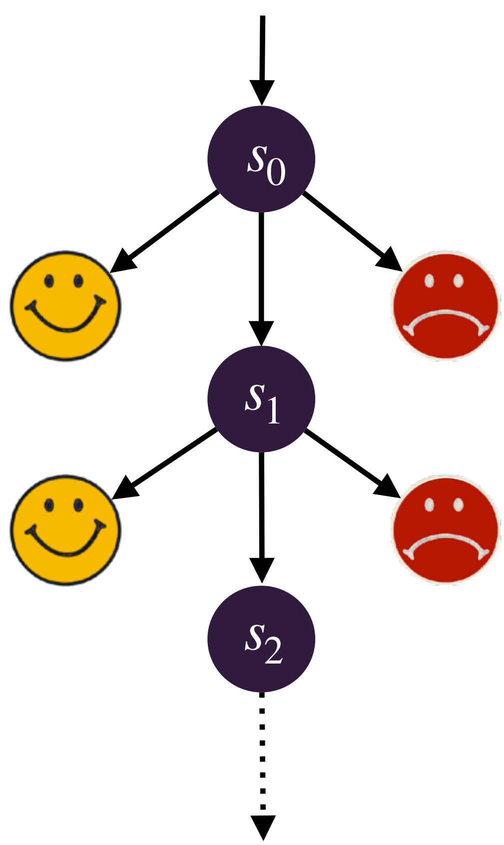

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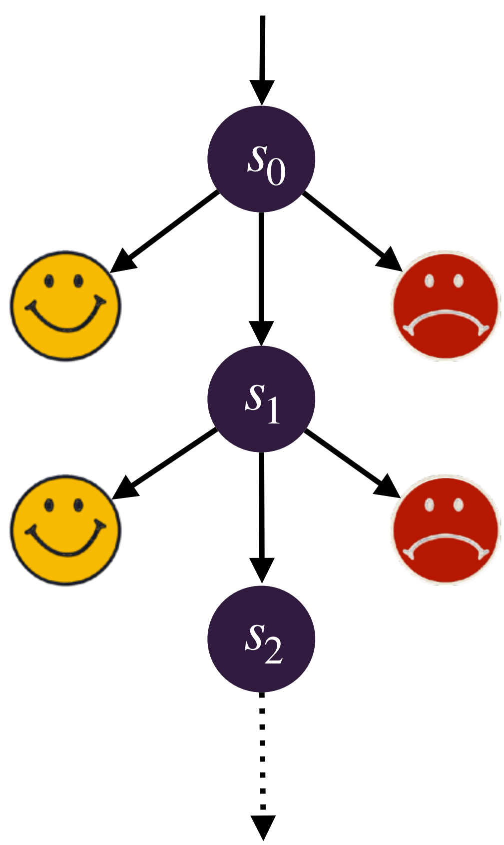

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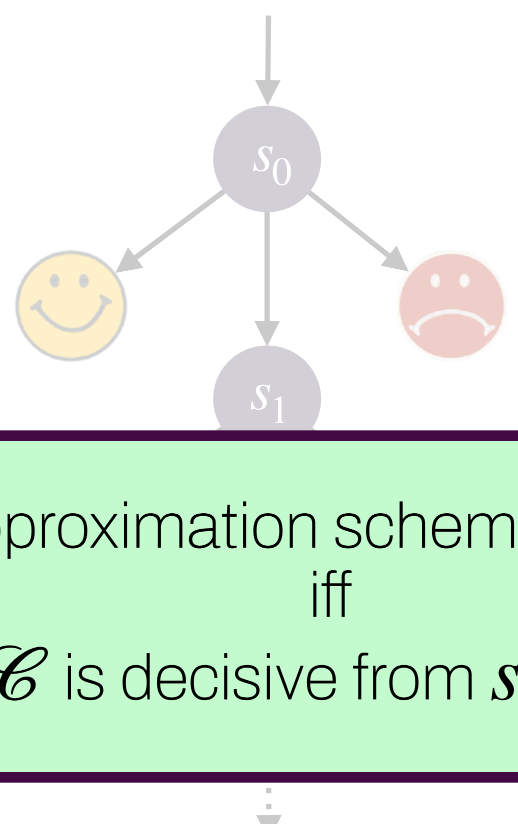

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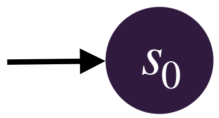
The approximation scheme converges
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. 🙂

[IN97] P. Iyer, M. Narasimha. Probabilistic lossy channel systems (TAPSOFT'97)
[ABM07] P.A. Abdulla, N. Ben Henda, R. Mayr. Decisive Markov chains (LMCS, 2007)

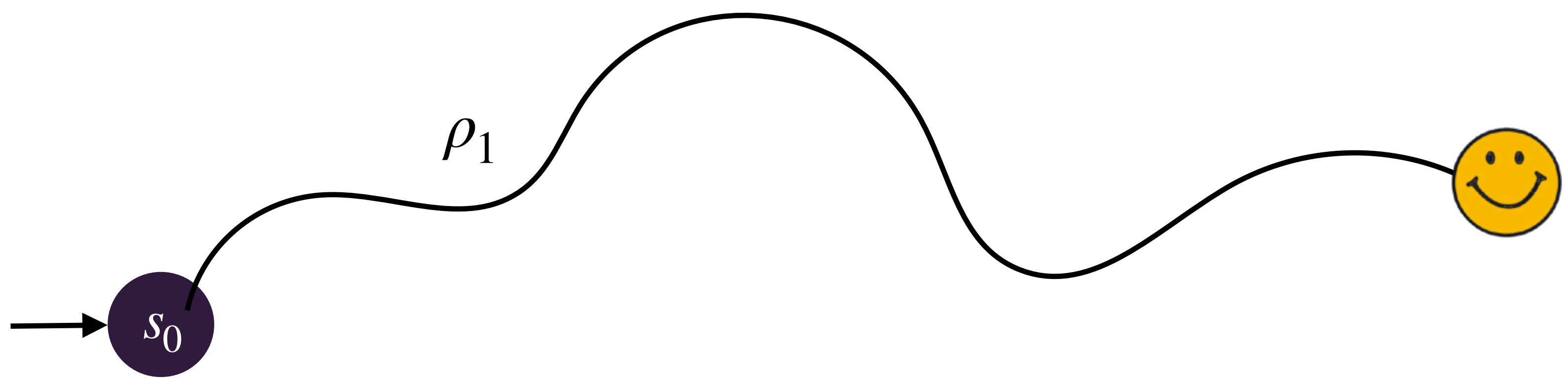
Statistical model-checking

Sample N paths



Statistical model-checking

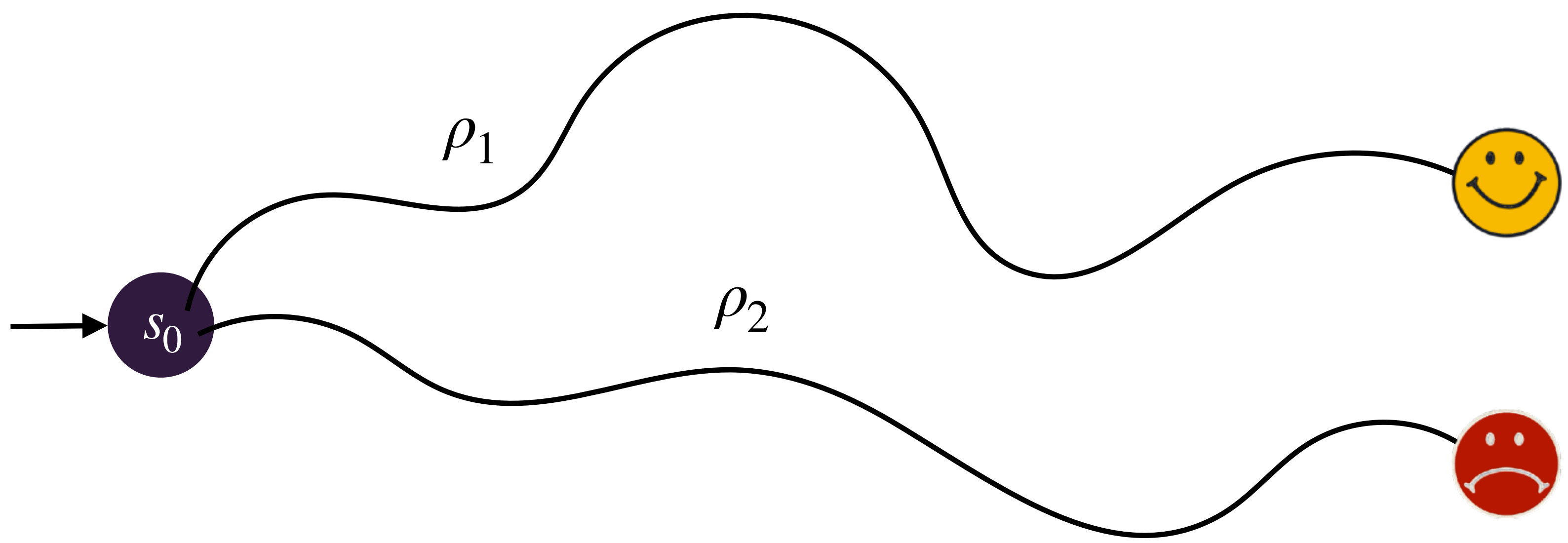
Sample N paths



$n_1 = 1$

Statistical model-checking

Sample N paths

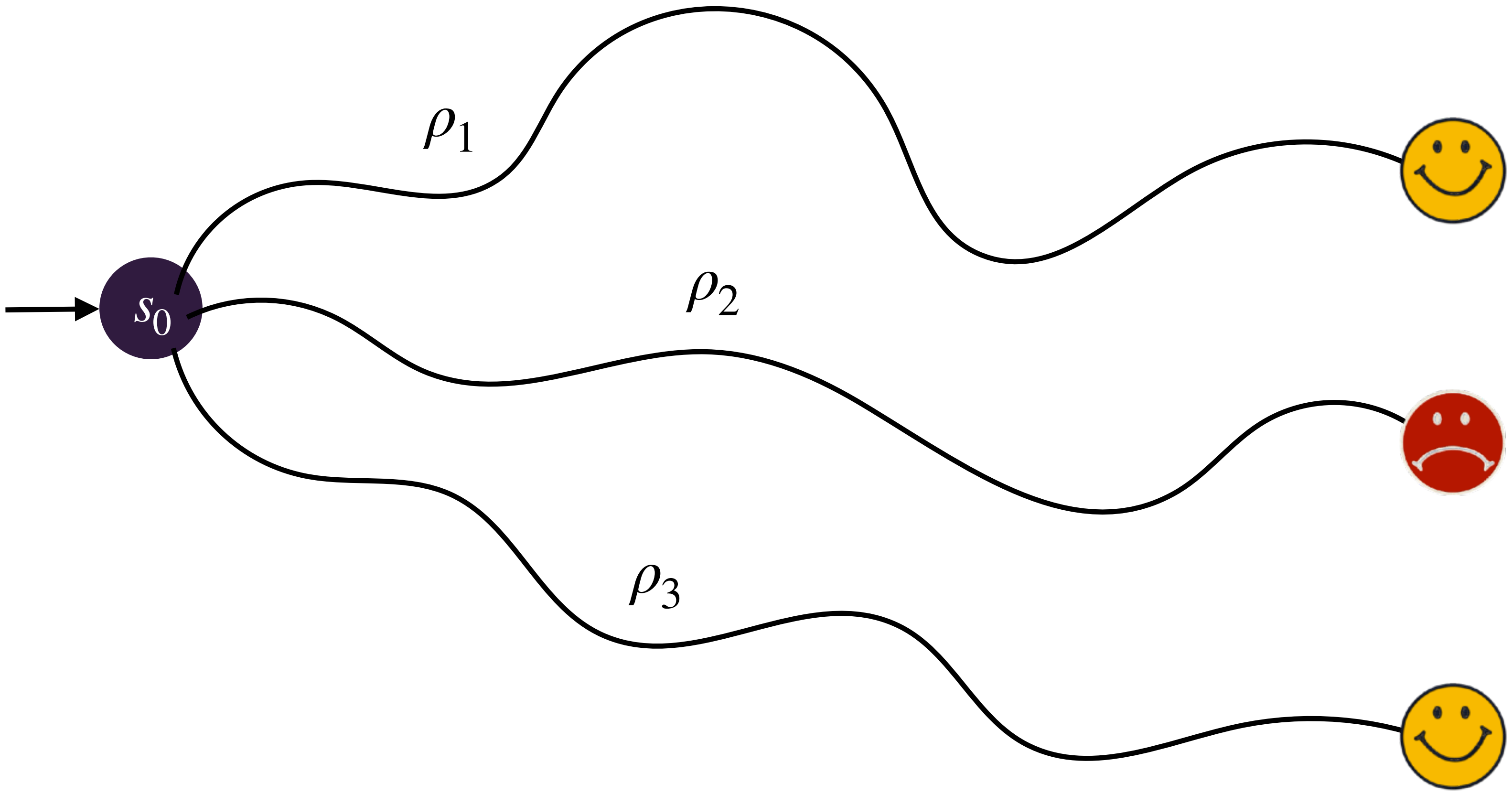


$n_1 = 1$

$n_2 = n_1$

Statistical model-checking

Sample N paths



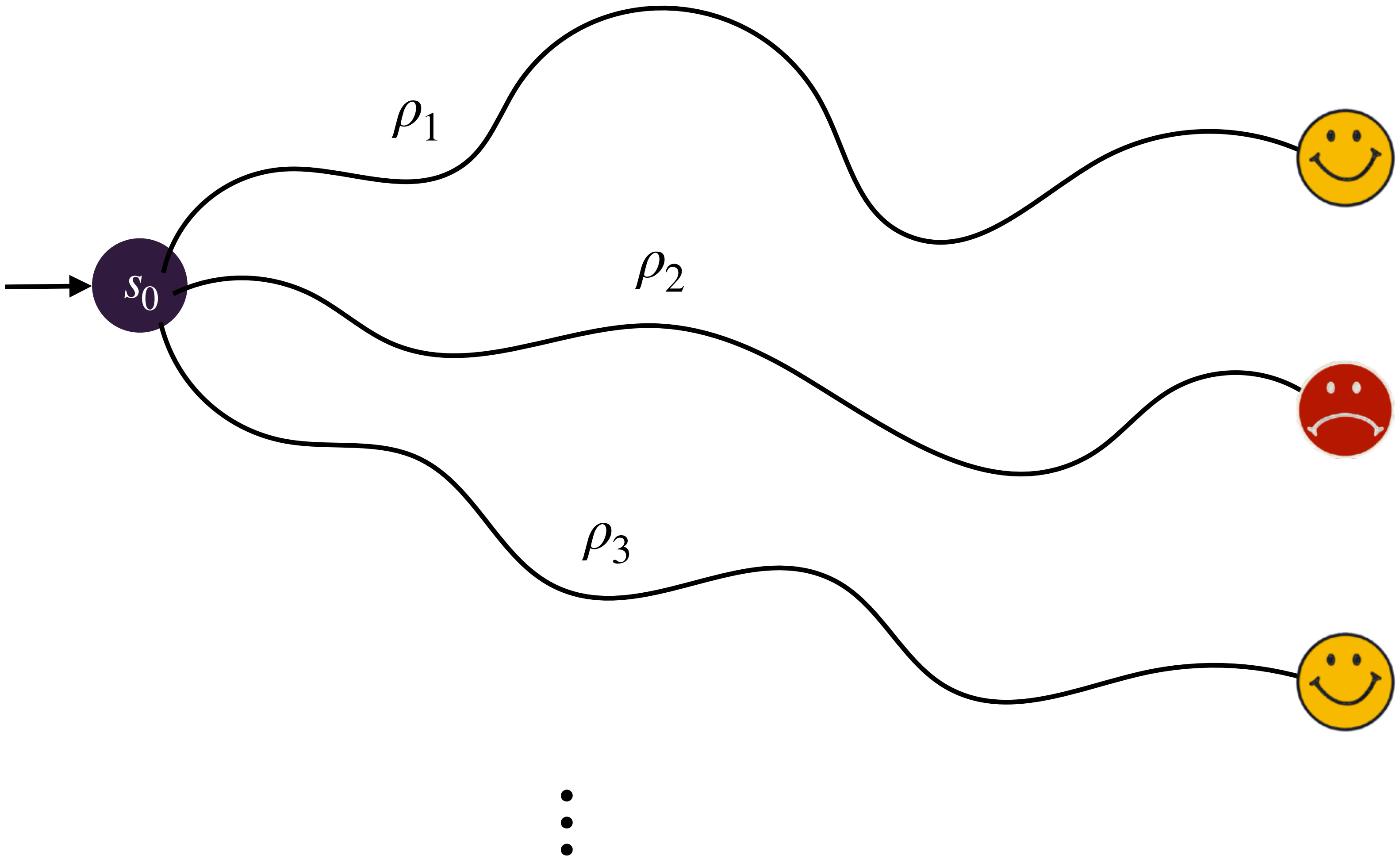
$n_1 = 1$

$n_2 = n_1$

$n_3 = n_2 + 1$

Statistical model-checking

Sample N paths



$n_1 = 1$

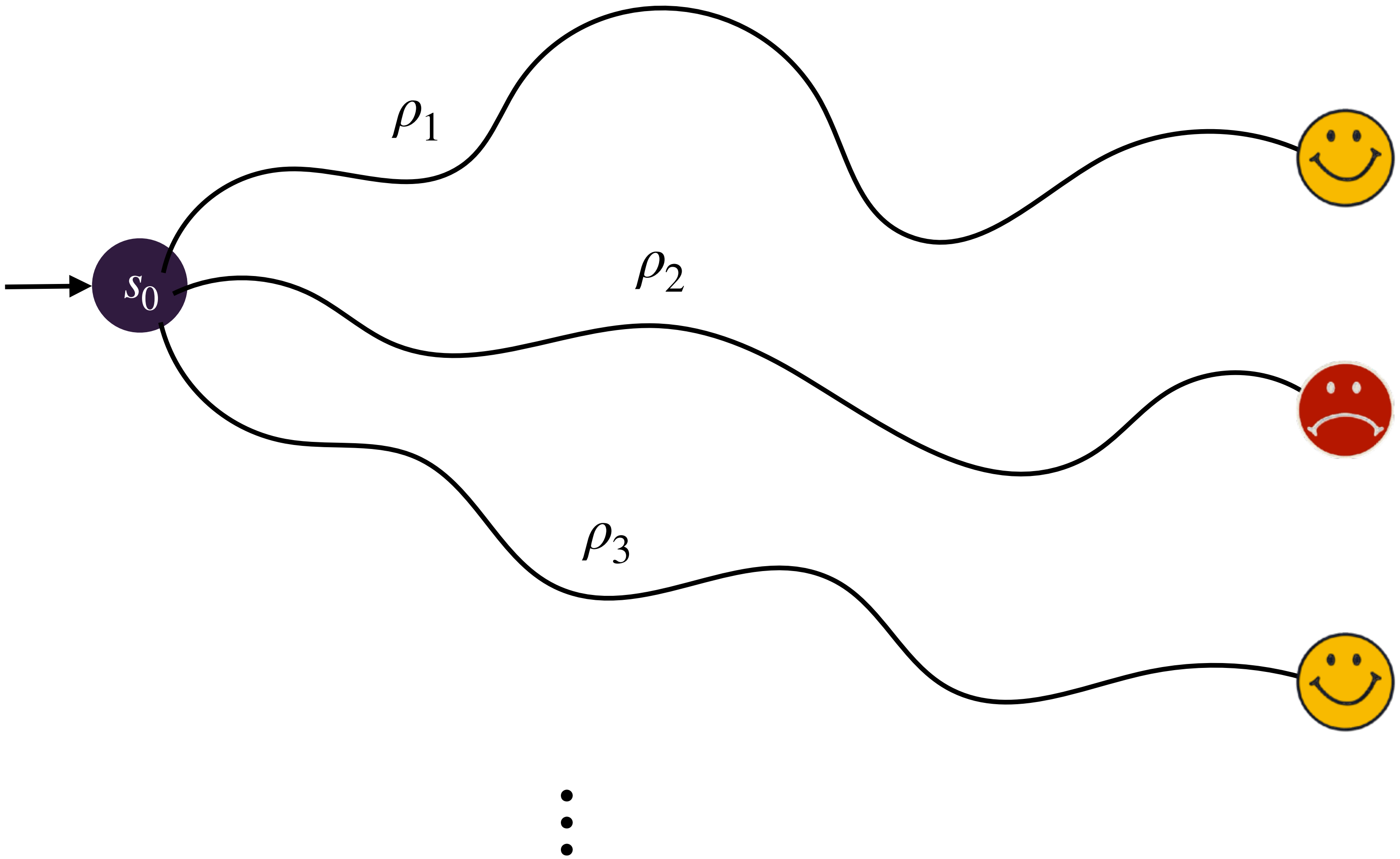
$n_2 = n_1$

$n_3 = n_2 + 1$

⋮

Statistical model-checking

Sample N paths



$n_1 = 1$

$n_2 = n_1$

$n_3 = n_2 + 1$

$\vdots$

Return $\frac{n_N}{N}$ + some confidence interval

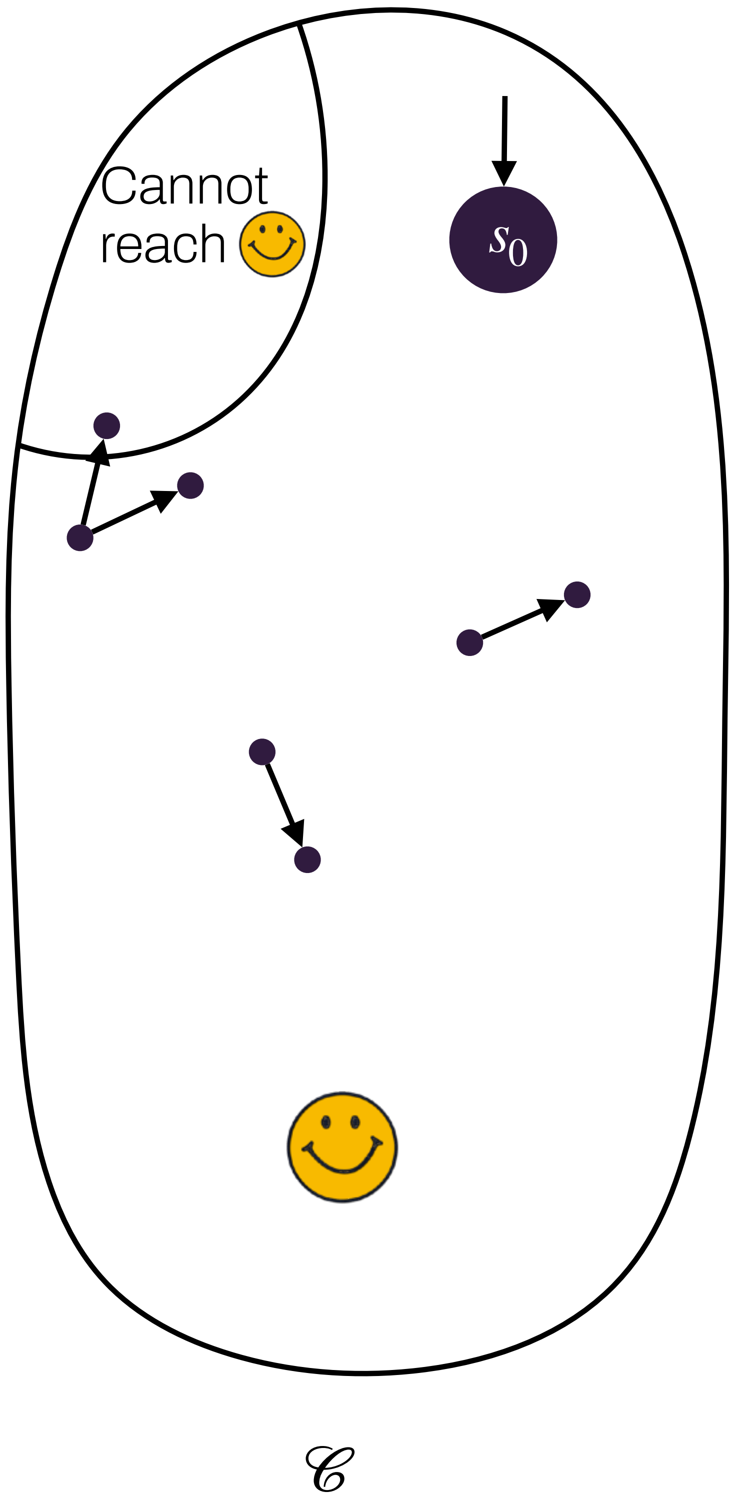
(in the best case)

Termination, efficiency and guarantees

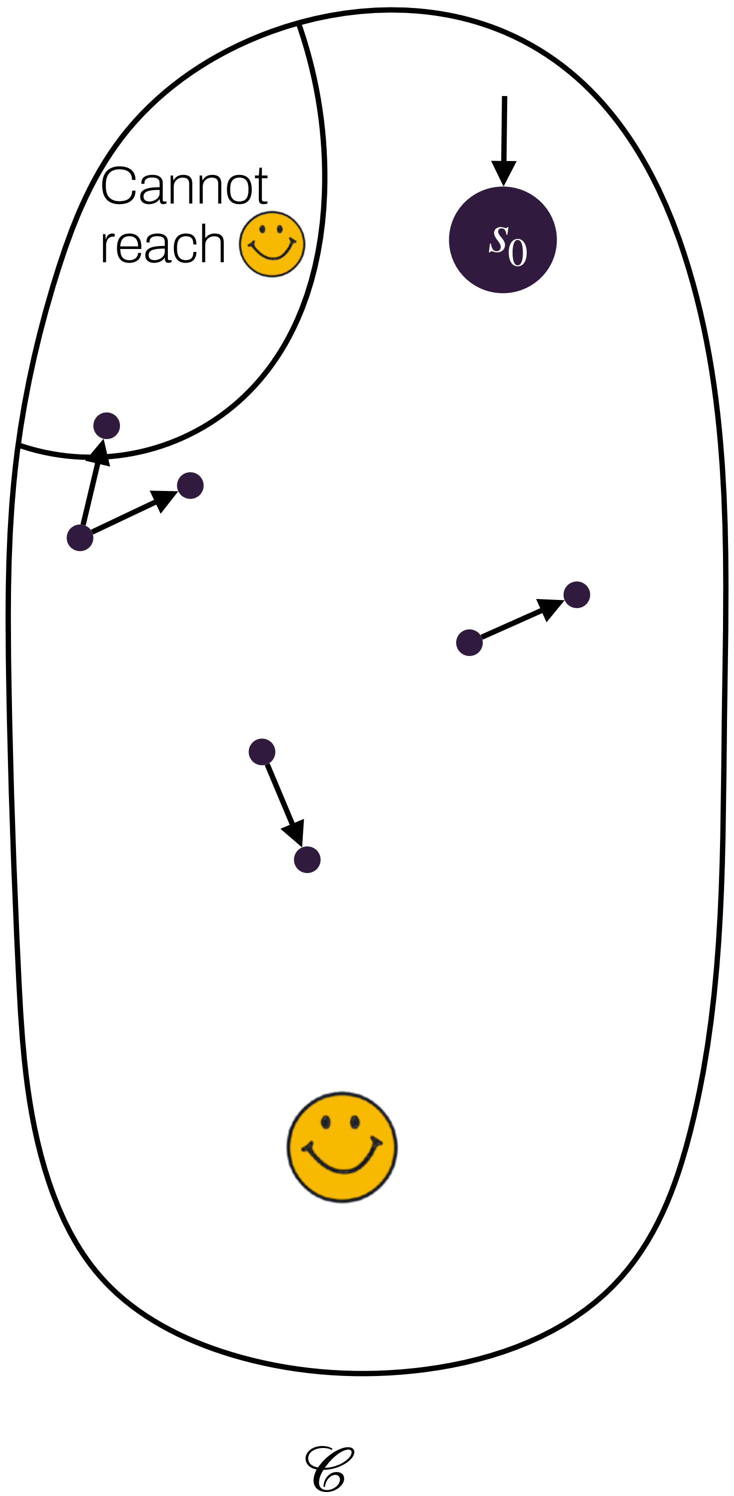
Termination (To our knowledge, never expressed like this)

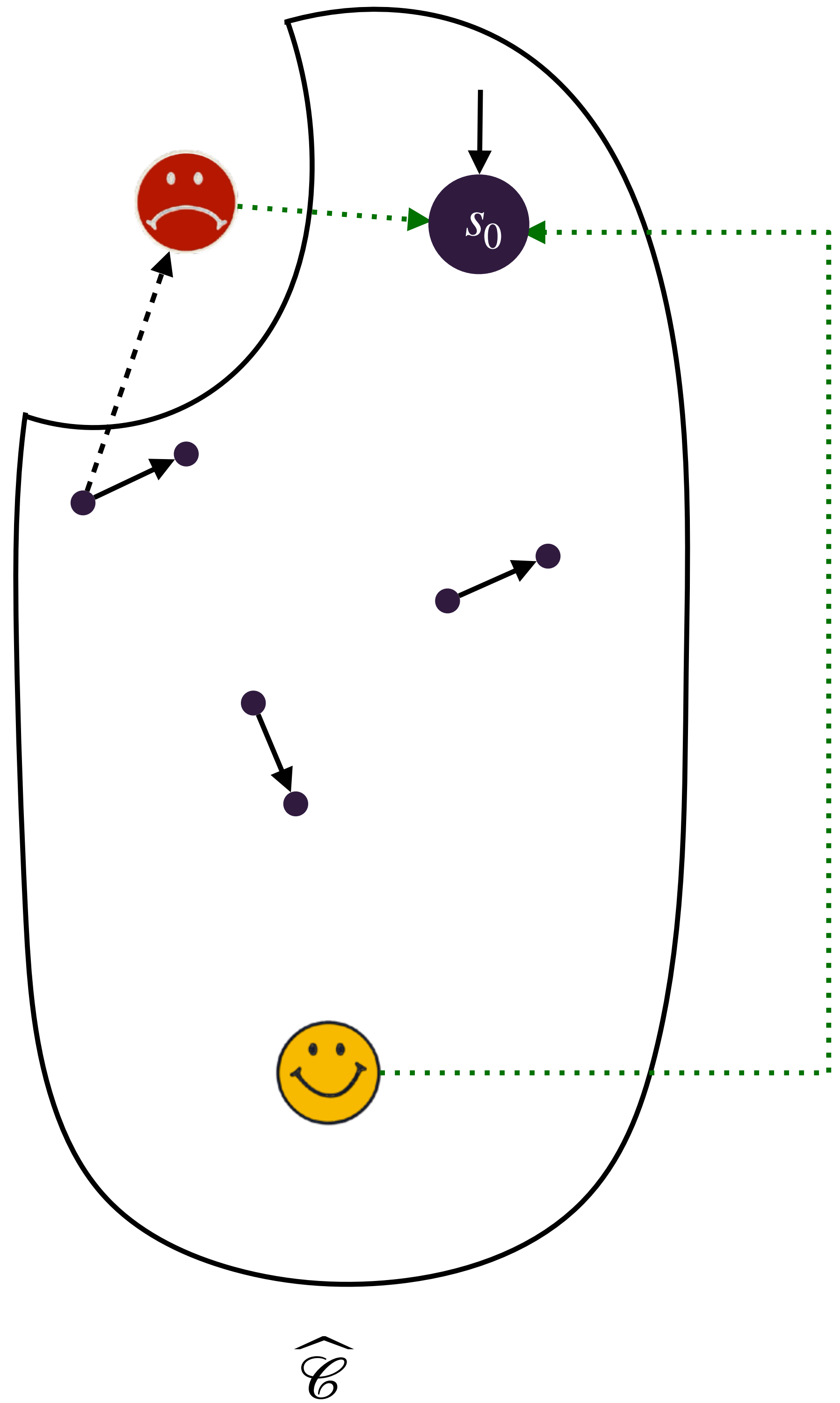
A sampled path starting at s_0 almost-surely hits  or 
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. 

Decisiveness *vs* recurrence

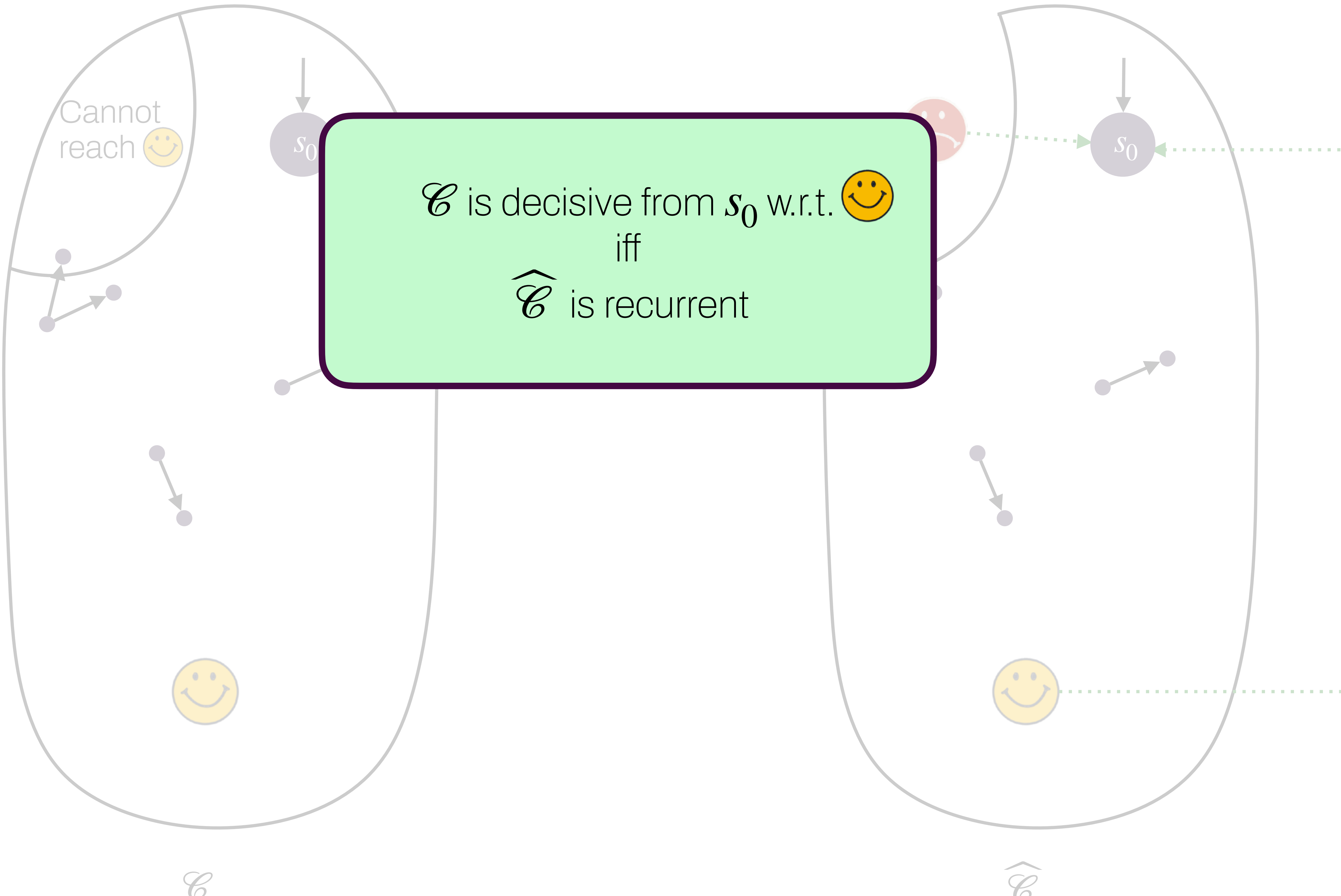


Decisiveness *vs* recurrence





Decisiveness *vs* recurrence

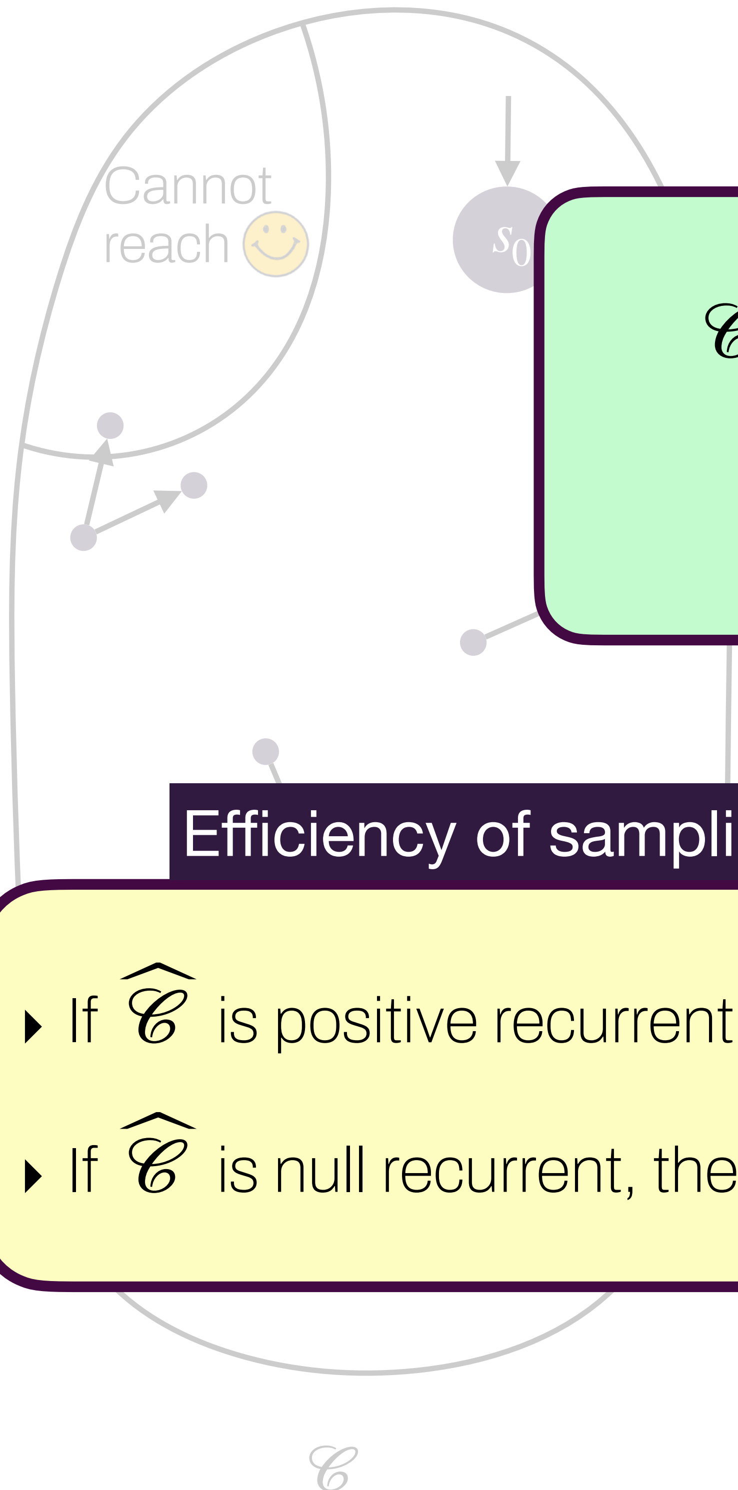


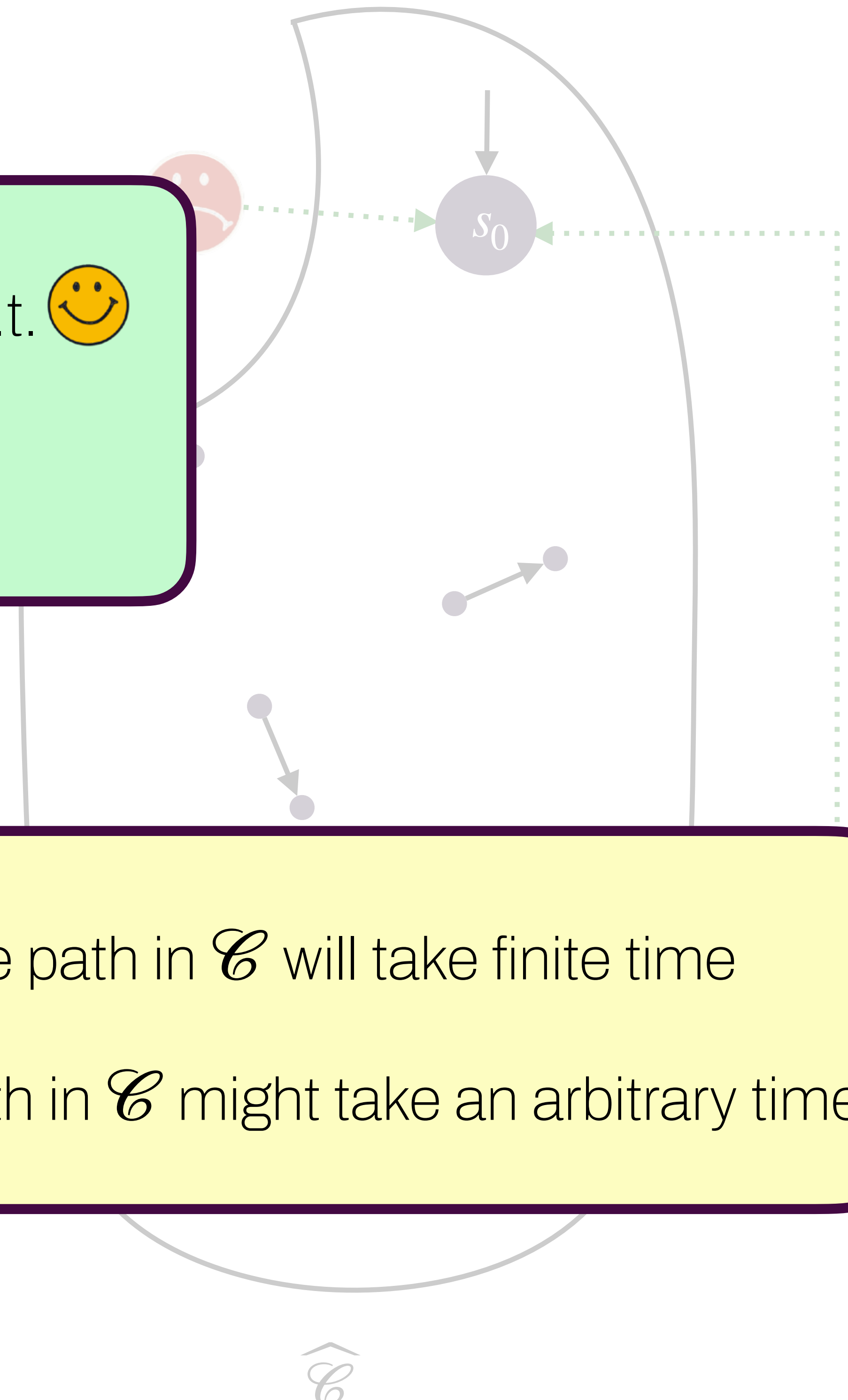
$\mathscr{C}$ is decisive from s_0 w.r.t. 🙂

iff

$\widehat{\mathscr{C}}$ is recurrent

Decisiveness *vs* recurrence



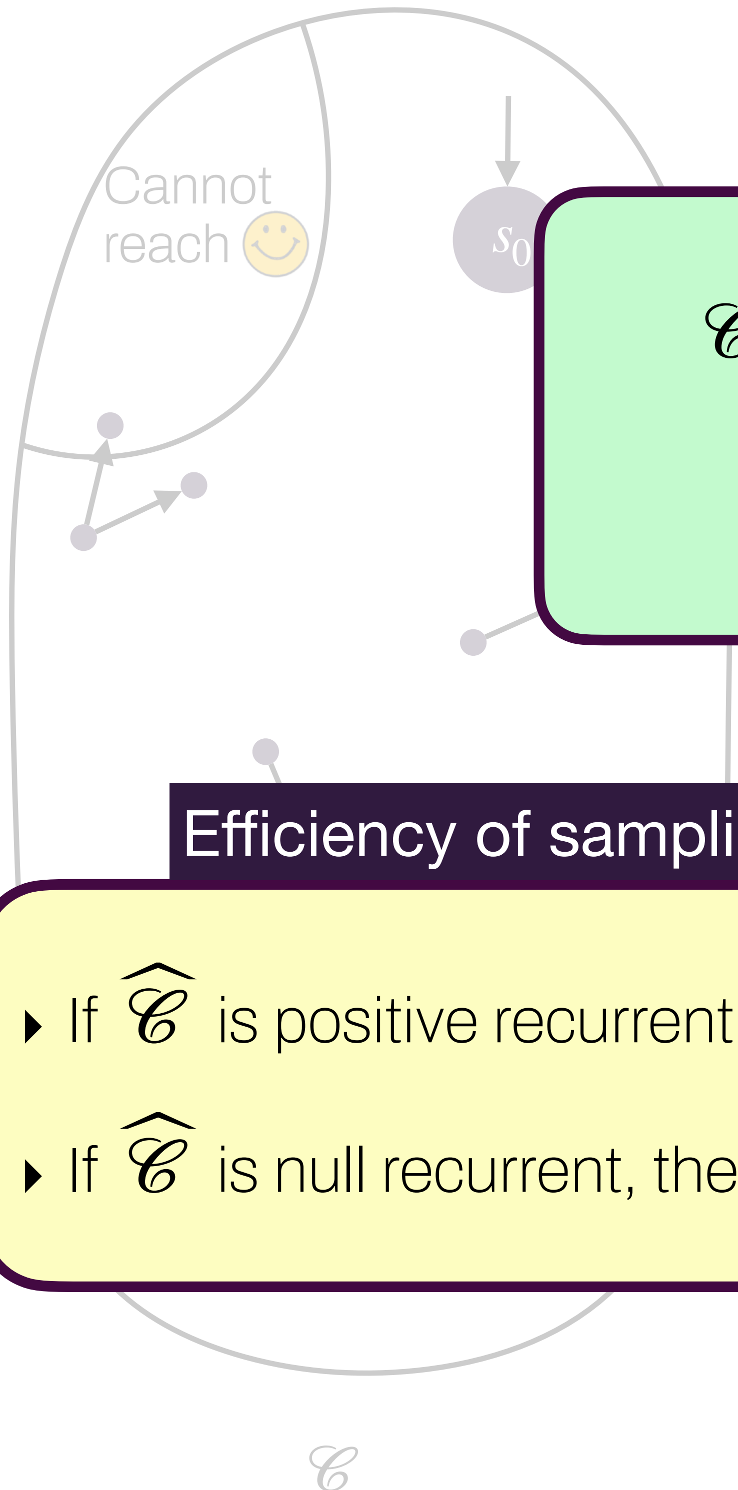


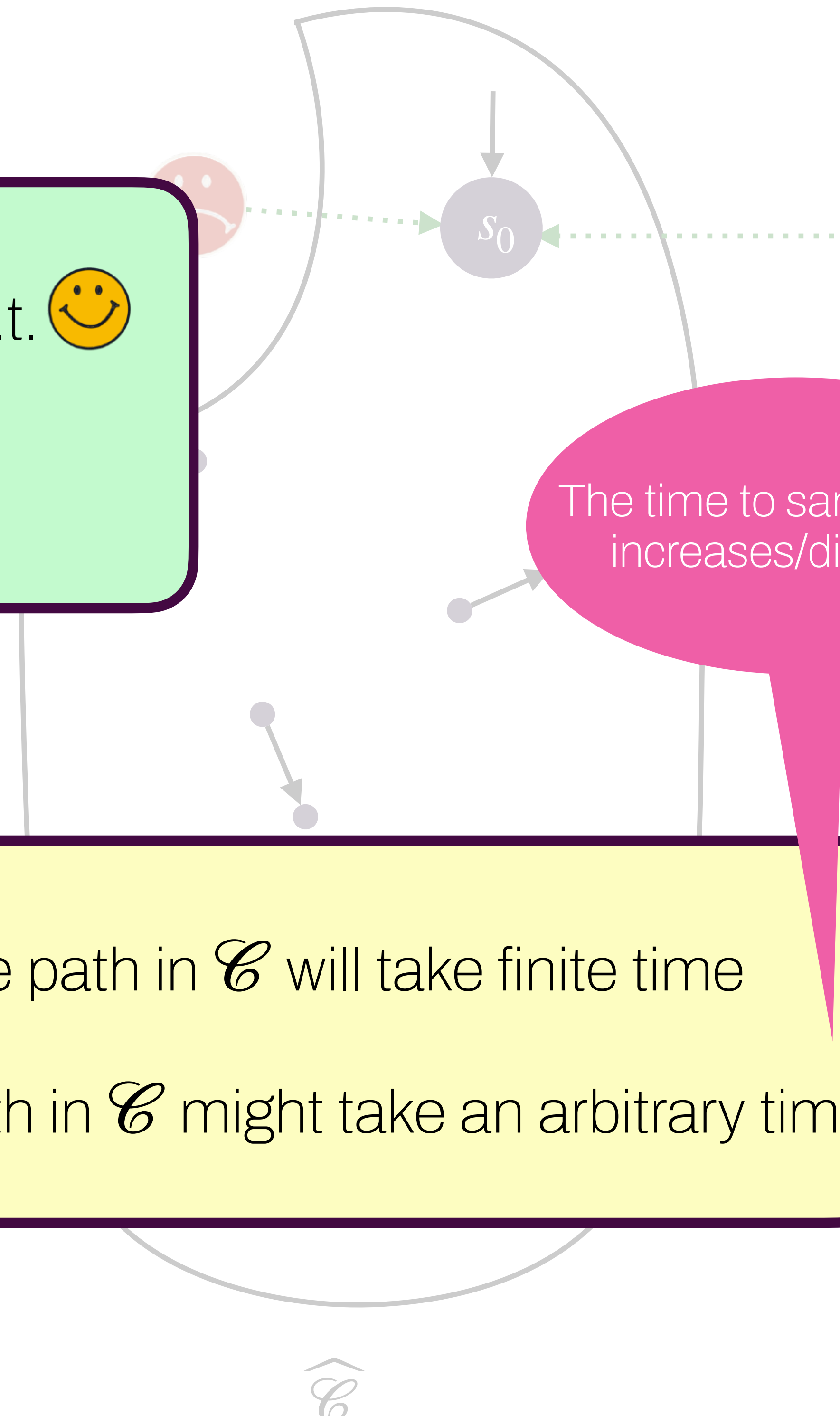
$\mathscr{C}$ is decisive from s_0 w.r.t. 😊
iff
$\widehat{\mathscr{C}}$ is recurrent

Efficiency of sampling

- If $\widehat{\mathscr{C}}$ is positive recurrent, then sampling a single path in $\mathscr{C}$ will take finite time
- If $\widehat{\mathscr{C}}$ is null recurrent, then sampling a single path in $\mathscr{C}$ might take an arbitrary time

Decisiveness *vs* recurrence





$\mathscr{C}$ is decisive from s_0 w.r.t. ☺
iff
$\widehat{\mathscr{C}}$ is recurrent

The time to sample even increases/diverges!

Efficiency of sampling

- If $\widehat{\mathscr{C}}$ is positive recurrent, then sampling a single path in $\mathscr{C}$ will take finite time
- If $\widehat{\mathscr{C}}$ is null recurrent, then sampling a single path in $\mathscr{C}$ might take an arbitrary time

Termination, efficiency and guarantees

Termination (To our knowledge, never expressed like this)

A sampled path starting at s_0 almost-surely hits  or 
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. 

+ efficiency if finite return time
(« $\widehat{\mathscr{C}}$ is positive recurrent »)

Termination, efficiency and guarantees

Termination

(To our knowledge, never expressed like this)

A sampled path starting at s_0 almost-surely hits ☺ or ☹
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. ☺

+ efficiency if finite return time
(« $\widehat{\mathscr{C}}$ is positive recurrent »)

Guarantees: Hoeffding's inequalities

Let $\varepsilon, \delta > 0$, let $N \geq \frac{8}{\varepsilon^2} \log\left(\frac{2}{\delta}\right)$. Then:

$$\mathbb{P}\left(\left| \frac{n_N}{N} - \mathbb{P}(\mathbf{F}\, ☺) \right| \geq \frac{\varepsilon}{2} \right) \leq \delta$$

Termination, efficiency and guarantees

Termination

(To our knowledge, never expressed like this)

A sampled path starting at s_0 almost-surely hits 🙂 or ☹️
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. 🙂

+ efficiency if finite return time
(« $\widehat{\mathscr{C}}$ is positive recurrent »)

Guarantees: Hoeffding's inequalities

Let $\varepsilon, \delta > 0$, let $N \geq \frac{8}{\varepsilon^2} \log\left(\frac{2}{\delta}\right)$. Then:

Empirical average

$$\mathbb{P}\left(\left| \frac{n_N}{N} - \mathbb{P}(\mathbf{F}\,🙂) \right| \geq \frac{\varepsilon}{2} \right) \leq \delta$$

Termination, efficiency and guarantees

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(To our knowledge, never expressed like this)

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Empirical average

Precision

Termination, efficiency and guarantees

Termination

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Let $\varepsilon, \delta > 0$, let $N \geq \frac{8}{\varepsilon^2} \log\left(\frac{2}{\delta}\right)$. Then:

$$\mathbb{P}\left(\left| \frac{n_N}{N} - \mathbb{P}(\mathbf{F}\ 🙂) \right| \geq \frac{\varepsilon}{2} \right) \leq \delta$$

Empirical average

Confidence level

Precision

Termination, efficiency and guarantees

Termination (To our knowledge, never expressed like this)

A sampled path starting at s_0 almost-surely hits 😊 or ☹️
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. 😊

+ efficiency if finite return time
(« $\widehat{\mathscr{C}}$ is positive recurrent »)

Guarantees: Hoeffding's inequalities

Let $\varepsilon, \delta > 0$, let $N \geq \frac{8}{\varepsilon^2} \log\left(\frac{2}{\delta}\right)$. Then:

$$\mathbb{P}\left(\left|\frac{n_N}{N} - \mathbb{P}(\mathbf{F}\,😊)\right| \geq \frac{\varepsilon}{2}\right) \leq \delta$$

Empirical average

Confidence level

Precision

$\left[\frac{n_N}{N} - \frac{\varepsilon}{2}; \frac{n_N}{N} + \frac{\varepsilon}{2}\right]$: confidence interval

Termination, efficiency and guarantees

Termination

(To our knowledge, never expressed like this)

A sampled path starting at s_0 almost-surely hits 🙂 or ☹️
iff
$\mathscr{C}$ is decisive from s_0 w.r.t. 🙂

+ efficiency if finite return time
(« $\mathscr{C}$ is positive recurrent »)

Guarantees: Hoeffding's inequalities

Let $\varepsilon, \delta > 0$ s.t. $N \geq \frac{8B^2}{\varepsilon^2} \log\left(\frac{2}{\delta}\right)$. Then:

B bound on the function

$$\mathbb{P}\left(\left|\frac{f_N}{N} - \mathbb{E}(f_{L,🙂})\right| \geq \frac{\varepsilon}{2}\right) \leq \delta$$

Empirical estimation

Value given by L for paths that stop at 🙂

$\left[\frac{f_N}{N} - \frac{\varepsilon}{2}; \frac{f_N}{N} + \frac{\varepsilon}{2}\right]$: confidence interval

What can we do for
non-decisive Markov chains??

Importance sampling [KH51]

- Originally used for rare events

Rare events problem

- Issue: rare events in $\mathscr{C}$

Rare-Event Problem for Statistical Model Checking

Problem Statement

- We want to estimate the probability of a rare event e occurring with probability close to 10^{-15}.
- We want a *confidence level* of 0.99.
- We are able to compute 10^9 trajectories.

Possible Outcomes

Number of occurrences of e	Probability	Confidence interval
0	$\approx 1 - 10^{-6}$	$[0, 7.03 \cdot 10^{-9}]$
1	$\leq 10^{-6}$	$[6.83 \cdot 10^{-10}, 1.69 \cdot 10^{-9}]$
$n > 1$	$\leq 10^{-12}$	$> 6.83 \cdot 10^{-10}$

[Bar14] B. Barbot. Acceleration for statistical model checking (PhD thesis)
[BHP12] B. Barbot, S. Haddad, C. Picaronny. Coupling and Importance Sampling for Statistical Model Checking (TACAS'12)

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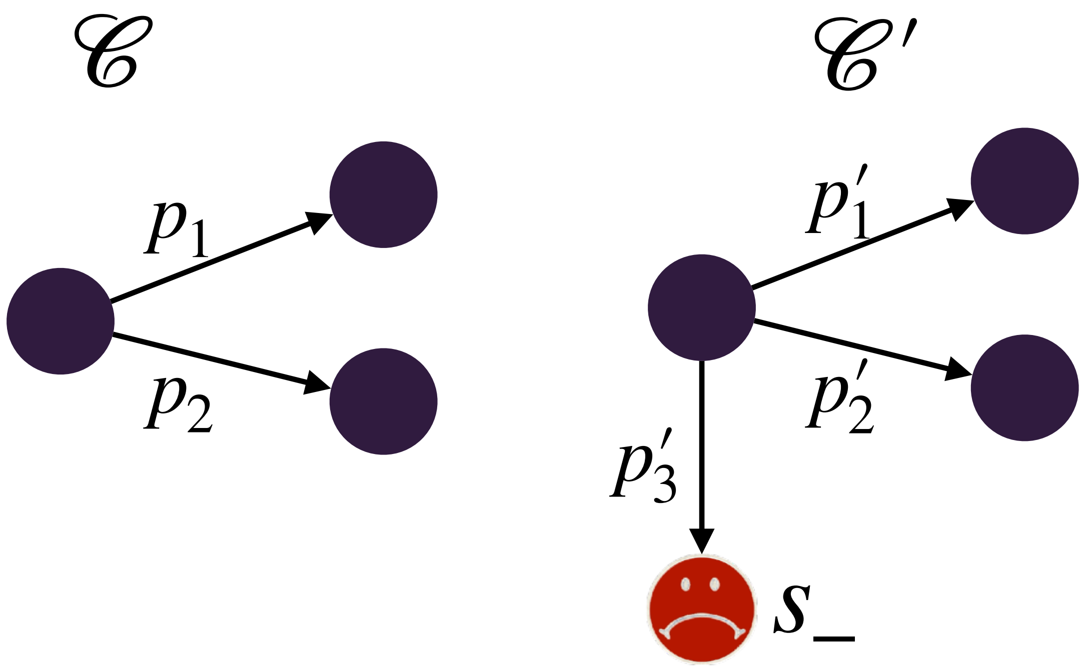
Importance sampling [KH51]

- Analyze a biased Markov chain $\mathscr{C}'$



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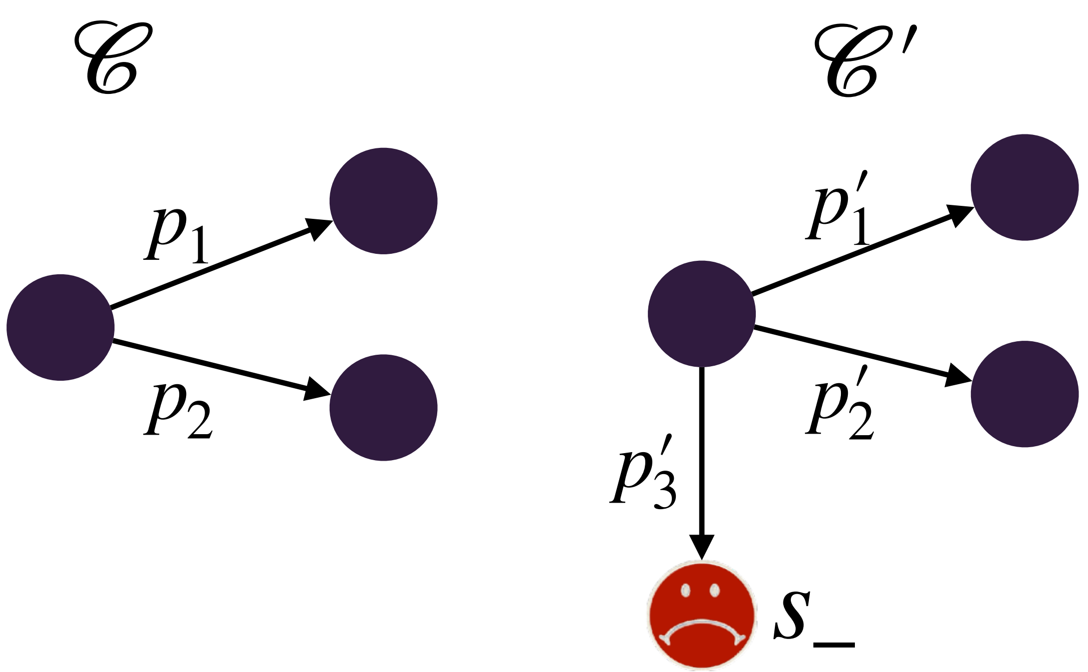

Correct the bias

$$\gamma(\rho) = \begin{cases} \dfrac{P(\rho)}{P'(\rho)} & \text{if } \rho \text{ ends in ☺} \\ 0 & \text{otherwise} \end{cases}$$

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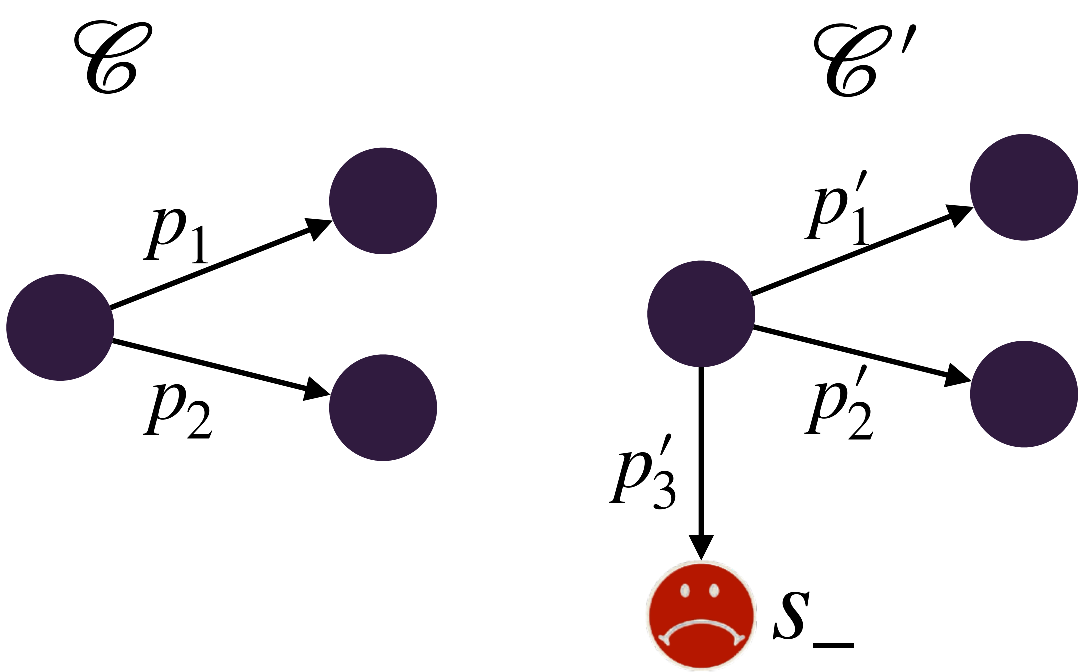

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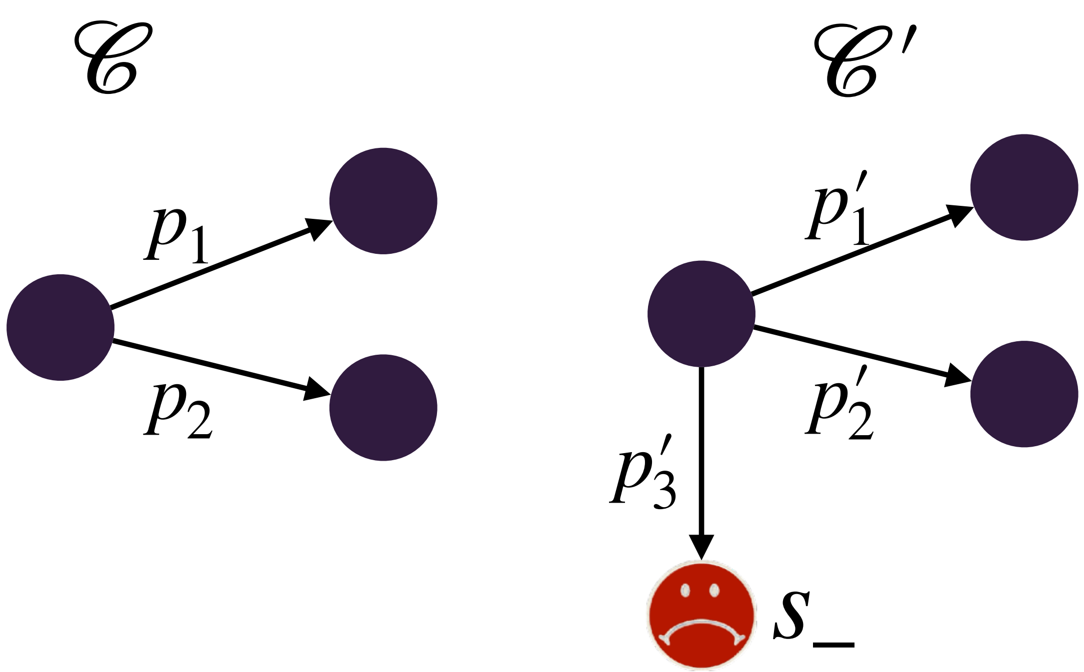

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It is sufficient to compute $\mathbb{E}_{\mathscr{C}'}(\gamma)$

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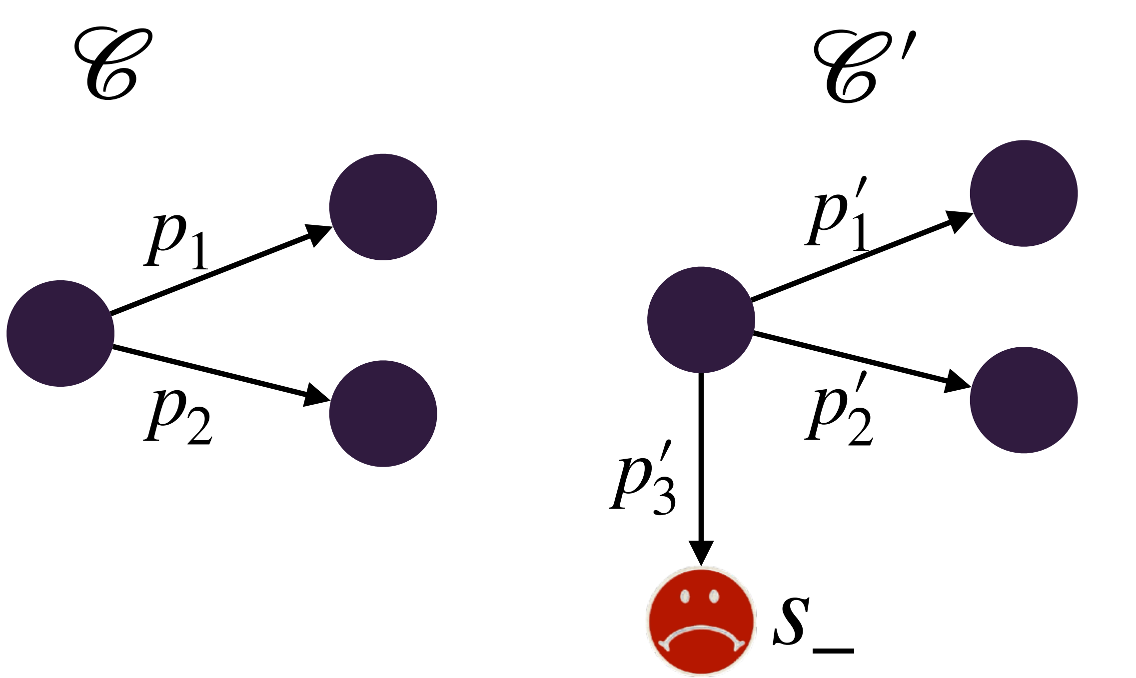

Correct the bias

$$\gamma(\rho) = \begin{cases} \dfrac{P(\rho)}{P'(\rho)} & \text{if } \rho \text{ ends in } \smiley \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}_{\mathscr{C}}(\mathbf{F}\,\smiley) = \mathbb{E}_{\mathscr{C}'}(\gamma)$$

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- Originally used for rare events
- Setting giving statistical guarantees [BHP12,Bar14]

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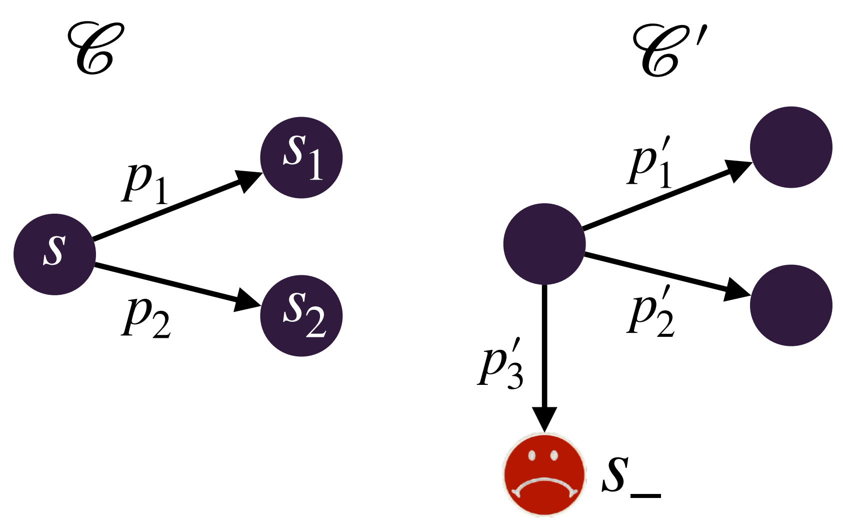
Properties of the biased Markov chain

$$\mathbb{P}_{\mathscr{C}}(\mathbf{F}\ 😊) = \mathbb{E}_{\mathscr{C}'}(\gamma)$$



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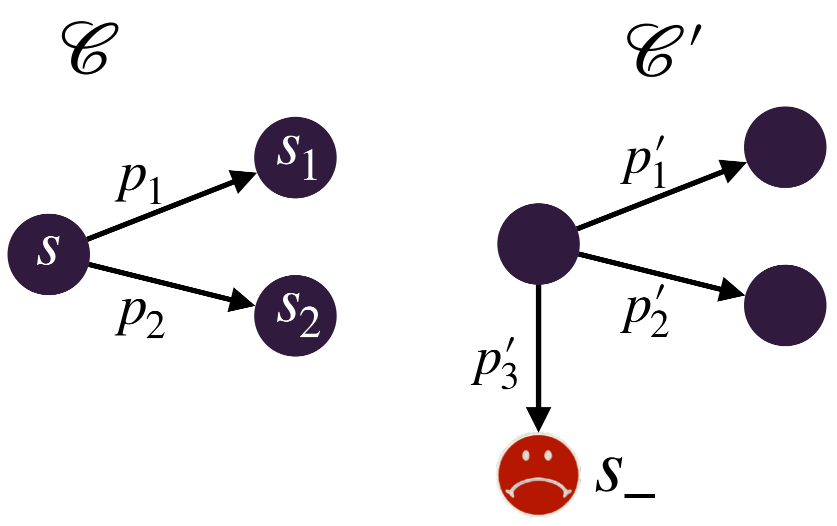
Properties of the biased Markov chain

$$\mathbb{P}_{\mathscr{C}}(\mathbf{F}\ ☺) = \mathbb{E}_{\mathscr{C}'}(\gamma)$$

- The analysis of $\mathscr{C}$ can be transferred to that of $\mathscr{C}'$, provided some conditions on $\mathscr{C}'$



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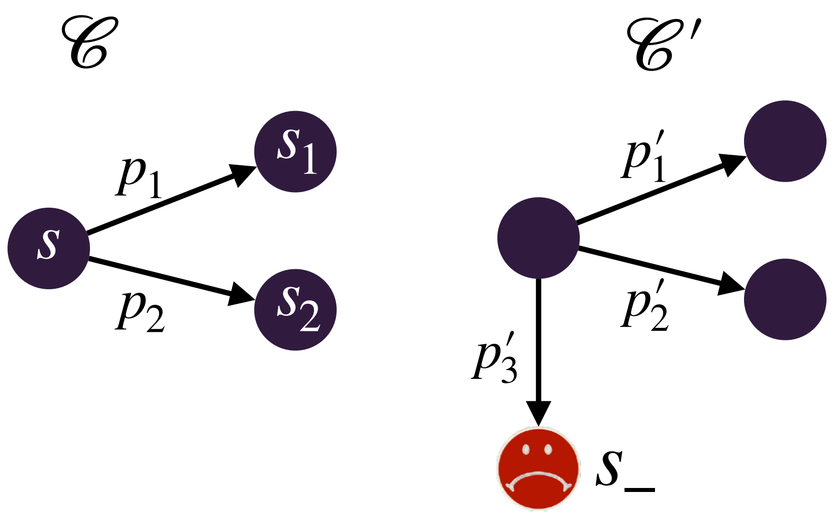
$$\mathbb{P}_{\mathscr{C}}(\mathbf{F}\ ☺) = \mathbb{E}_{\mathscr{C}'}(\gamma)$$

- The analysis of $\mathscr{C}$ can be transferred to that of $\mathscr{C}'$, provided some conditions on $\mathscr{C}'$
  - Decisiveness of $\mathscr{C}'$ is required for both approx. and estim. methods



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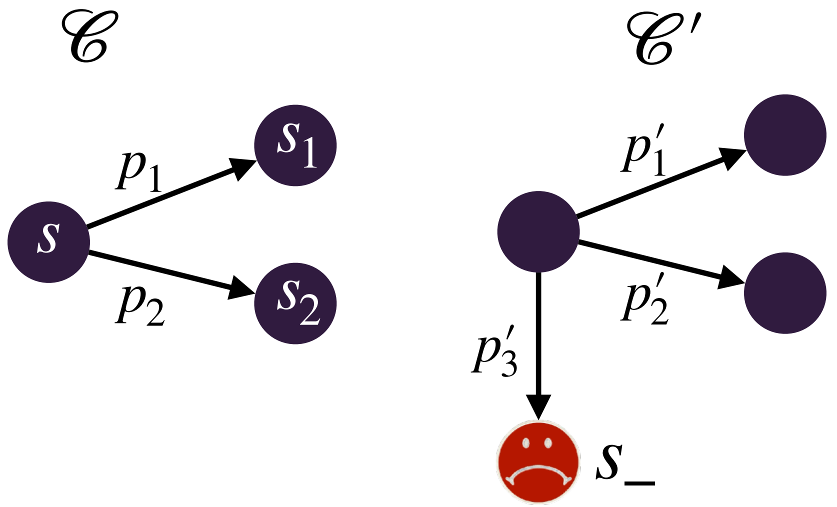
  - Boundedness of γ is required as well



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Properties of the biased Markov chain

Define $\mu(s)$ as $\mathbb{P}^{s}_{\mathscr{C}}(\mathbf{F} \smiley)$

$$\mathbb{P}_{\mathscr{C}}(\mathbf{F} \smiley) = \mathbb{E}_{\mathscr{C}'}(\gamma)$$

- The analysis of $\mathscr{C}$ can be transferred to that of $\mathscr{C}'$, provided some conditions on $\mathscr{C}'$
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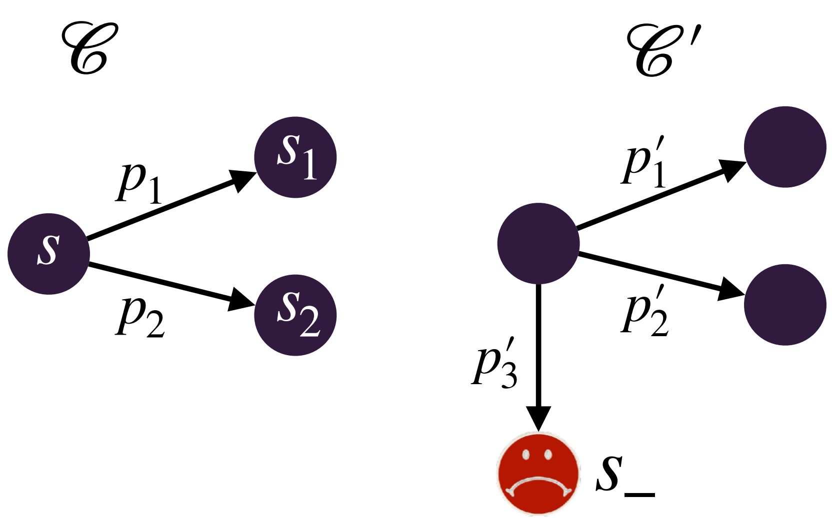

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Properties of the biased Markov chain

Define $\mu(s)$ as $\mathbb{P}^{s}_{\mathscr{C}}(\mathbf{F}\ \text{☺})$

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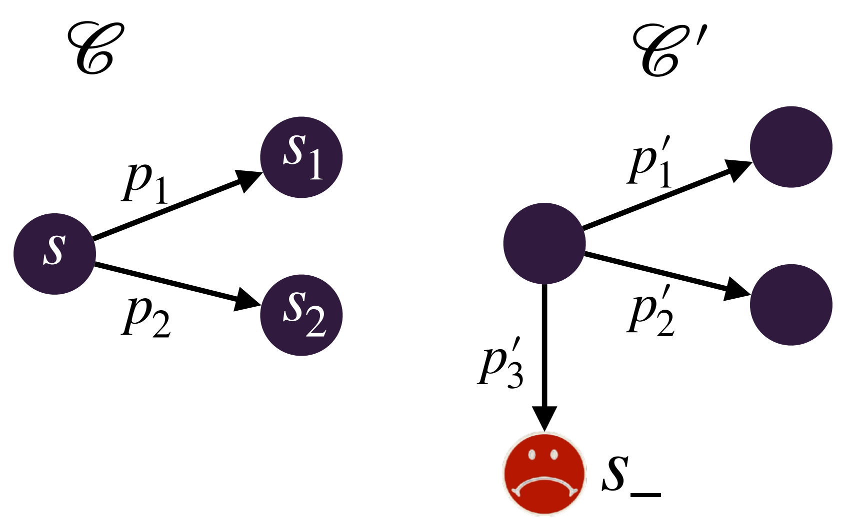
- The analysis of $\mathscr{C}$ can be transferred to that of $\mathscr{C}'$, provided some conditions on $\mathscr{C}'$
  - Decisiveness of $\mathscr{C}'$ is required for both approx. and estim. methods
  - Boundedness of γ is required as well

There is a best choice: $p'_i = \dfrac{\mu(s_i)}{\mu(s)} \cdot p_i$

- The r.v. in $\mathscr{C}'$ takes value $\mu(s)$
- One needs to know μ!



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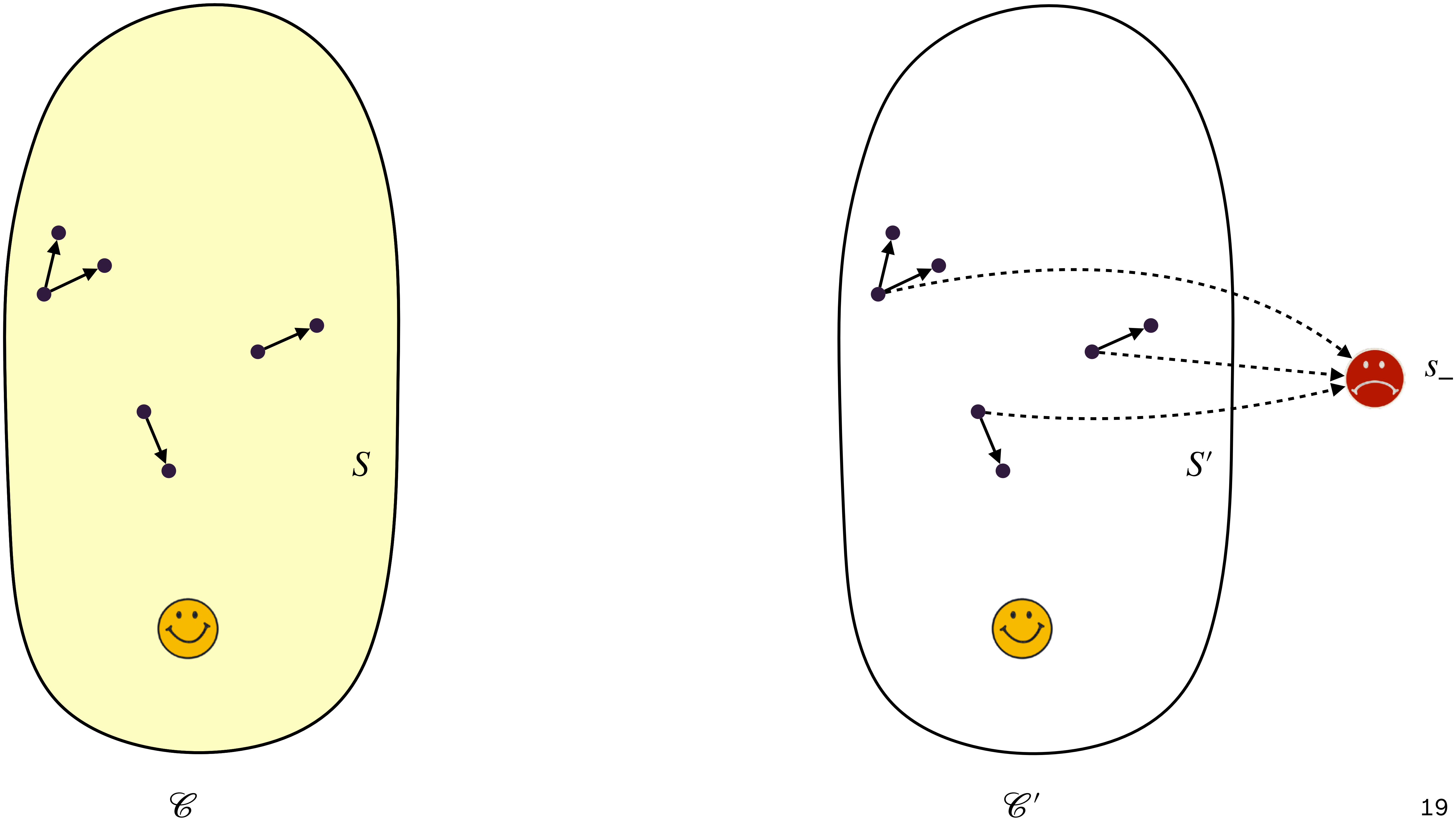
Importance sampling via an abstraction



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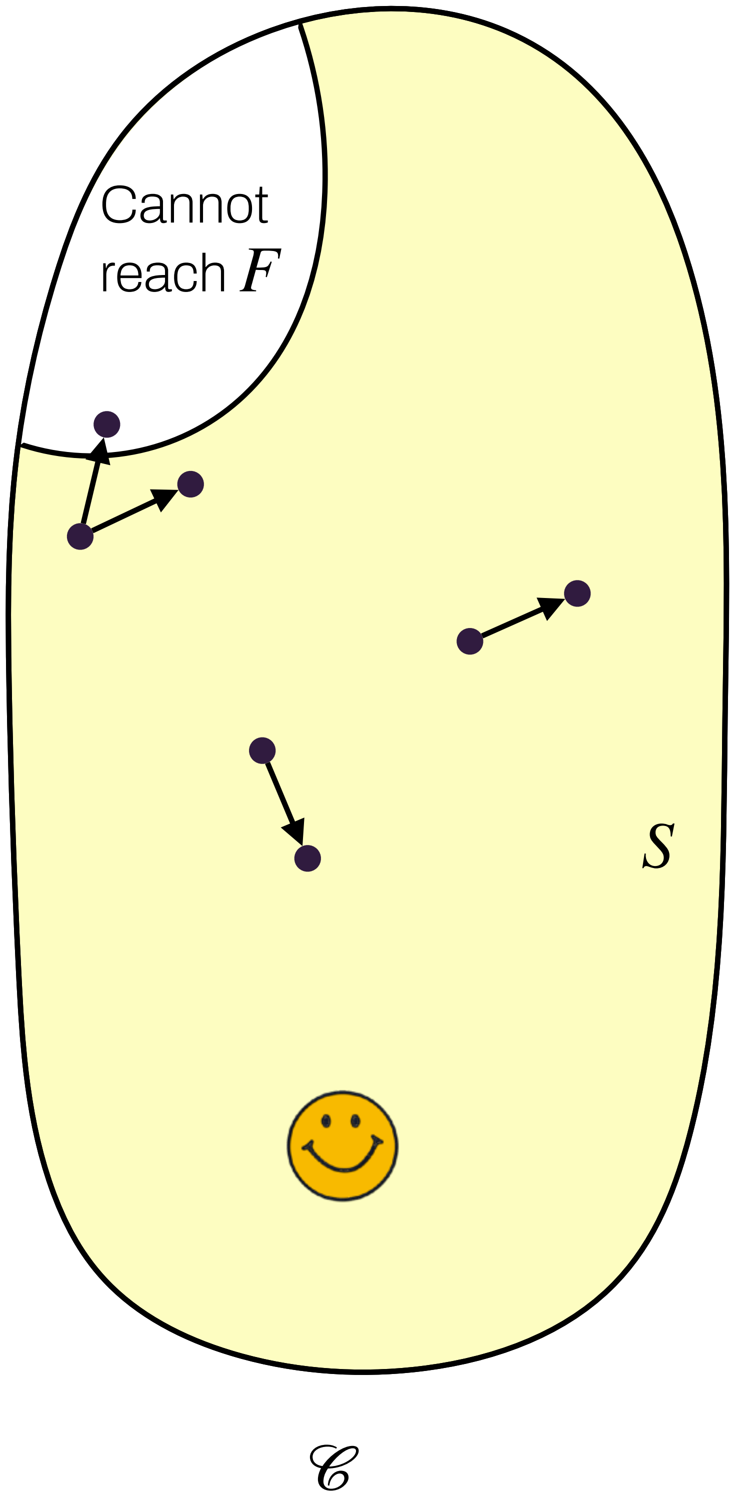

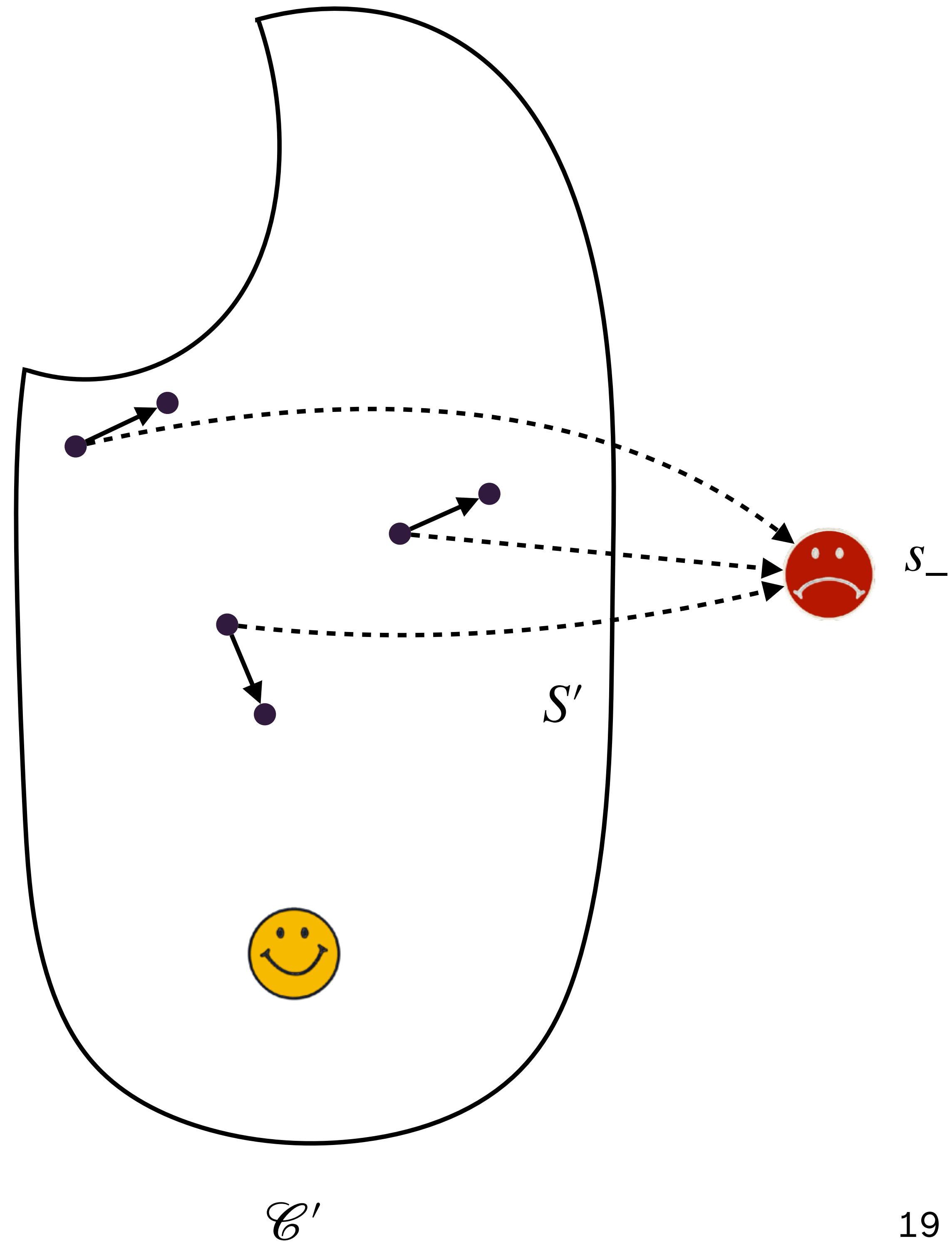

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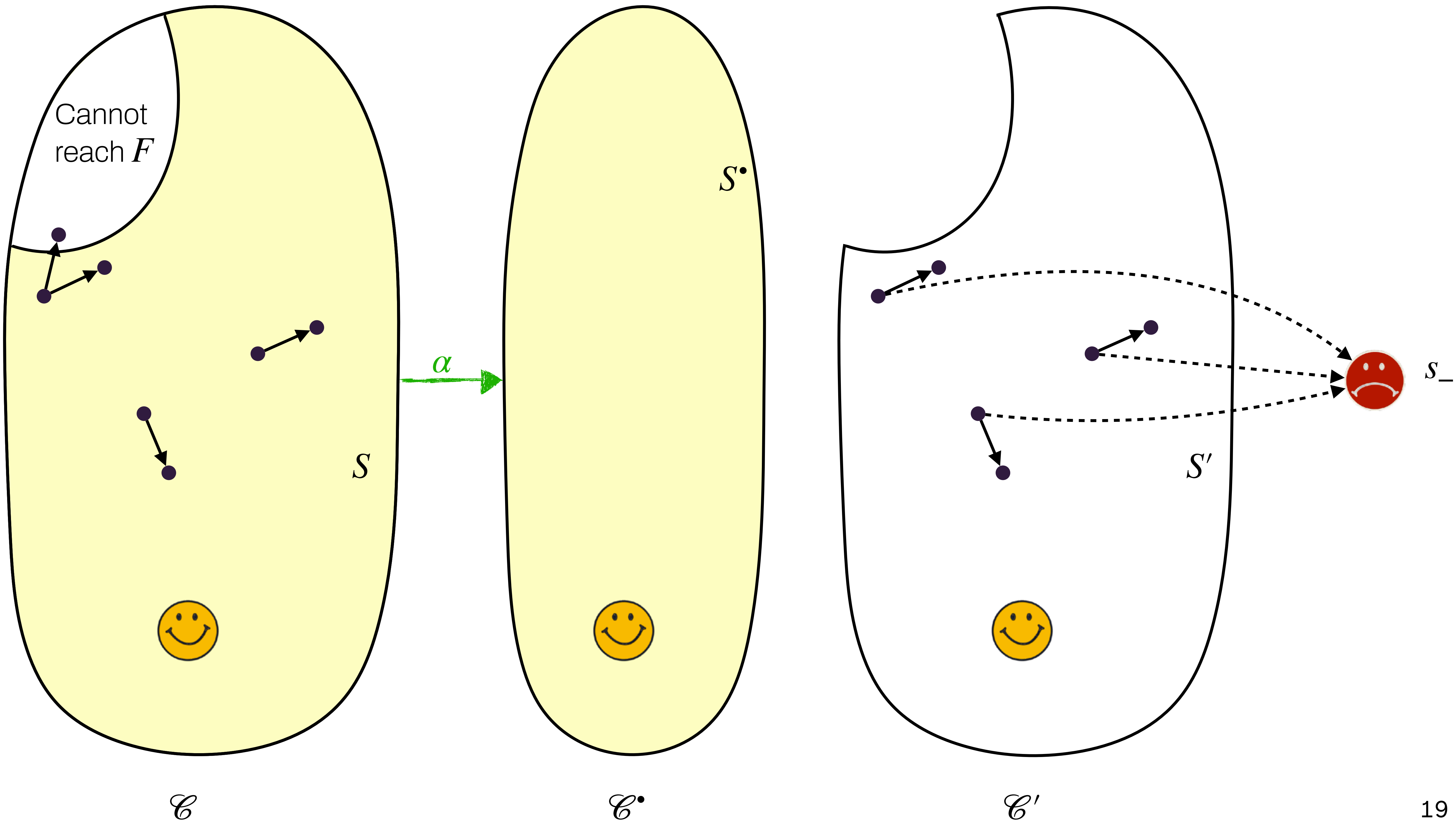
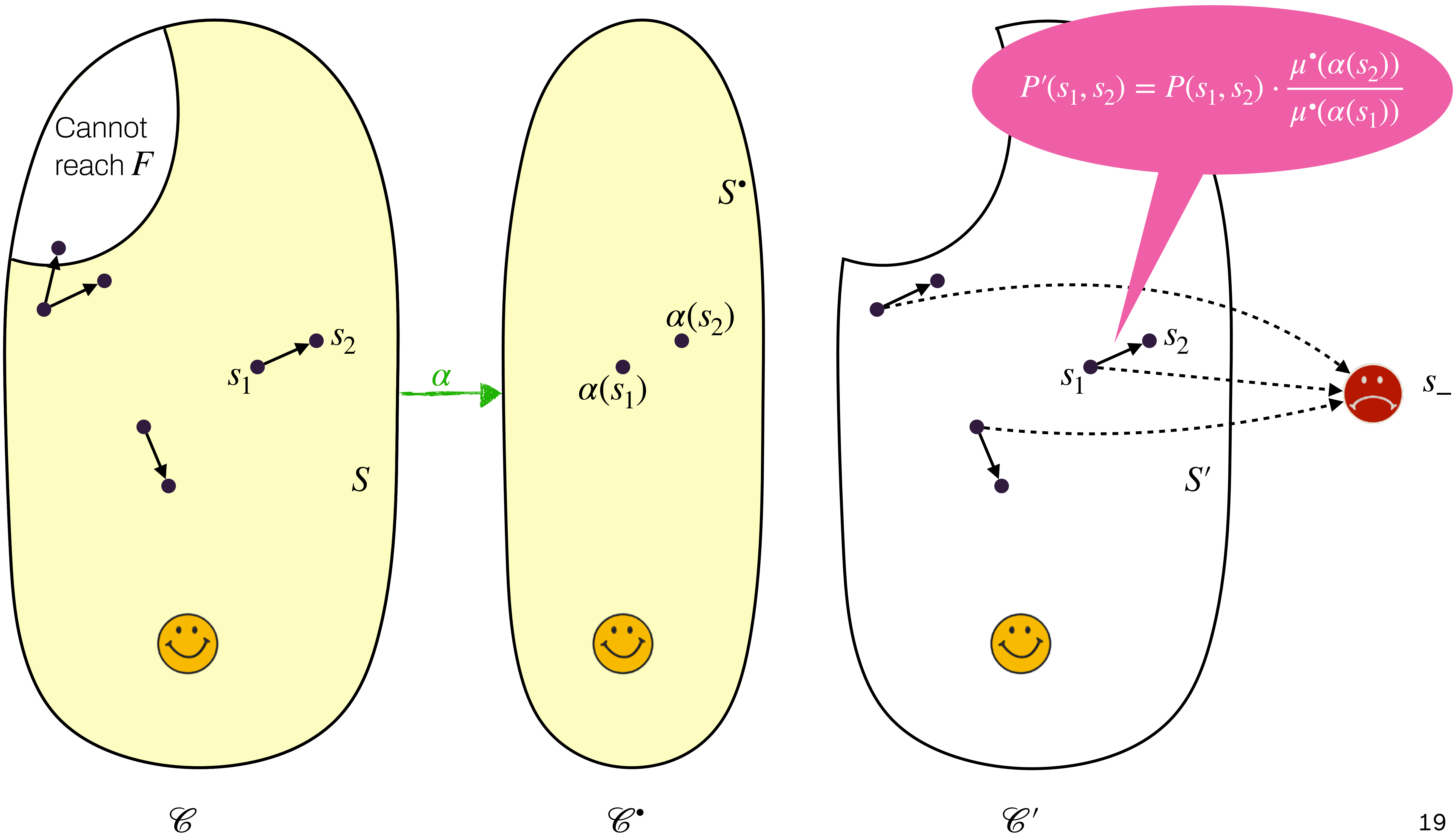


Importance sampling via an abstraction



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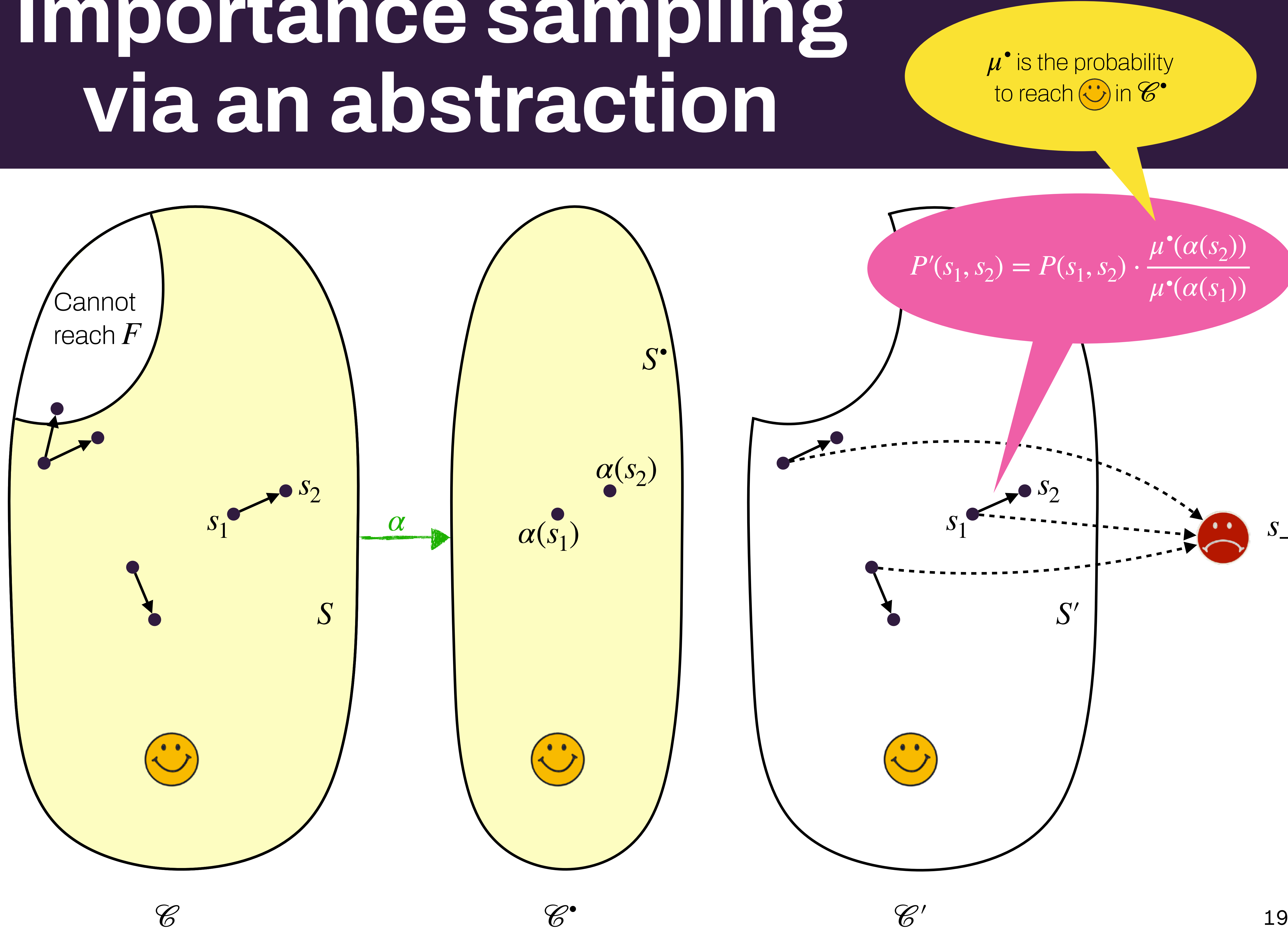

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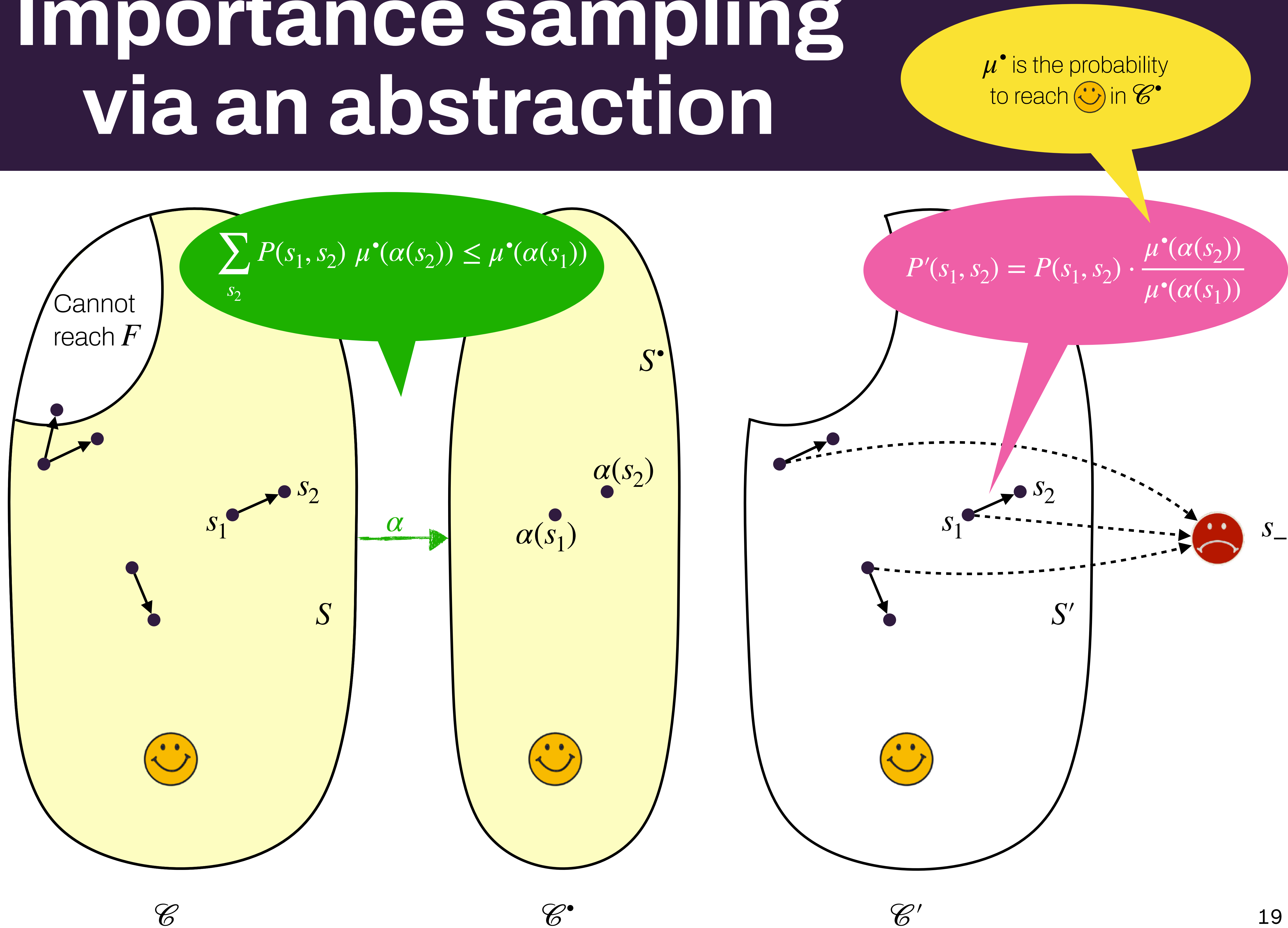

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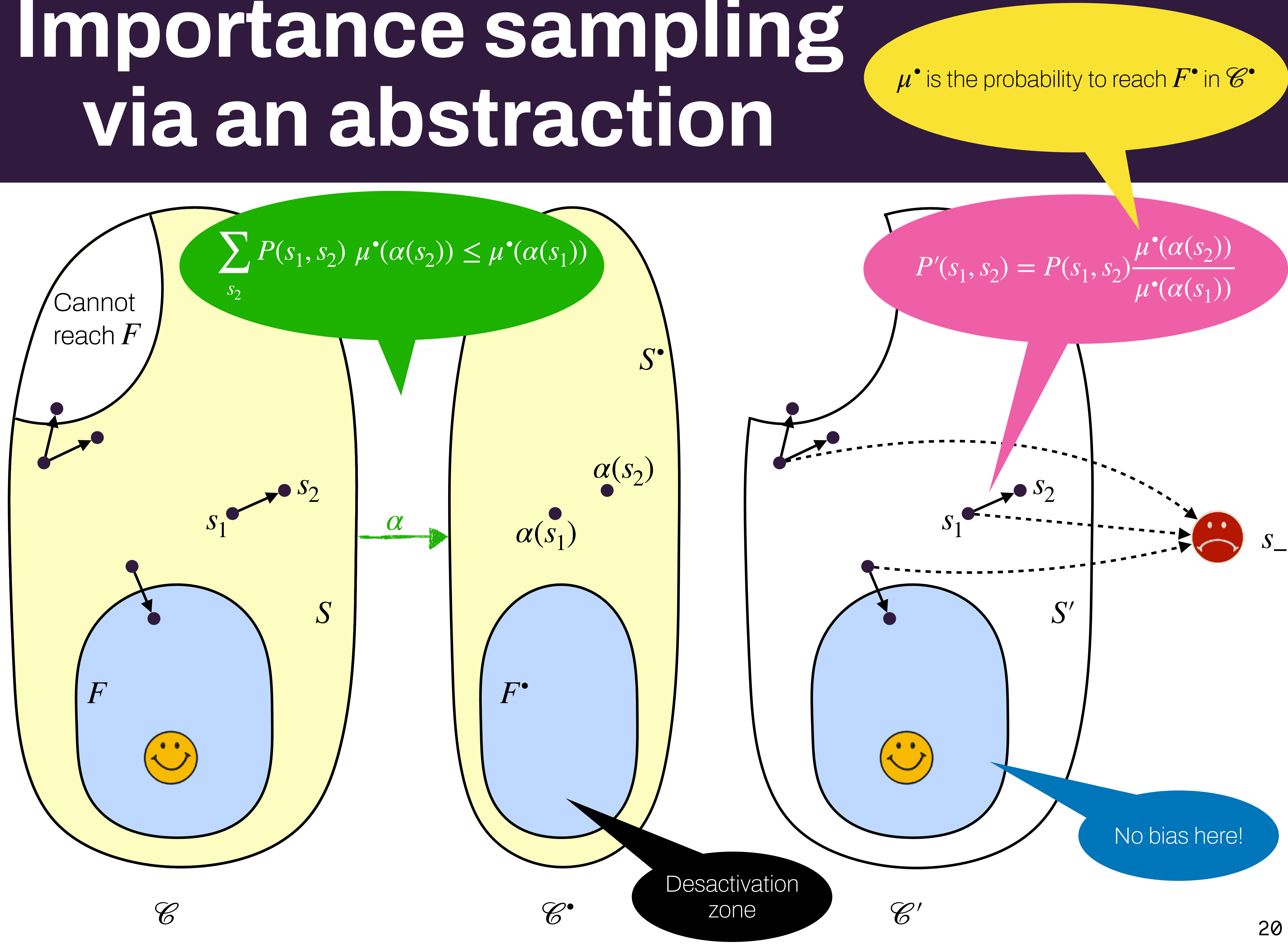
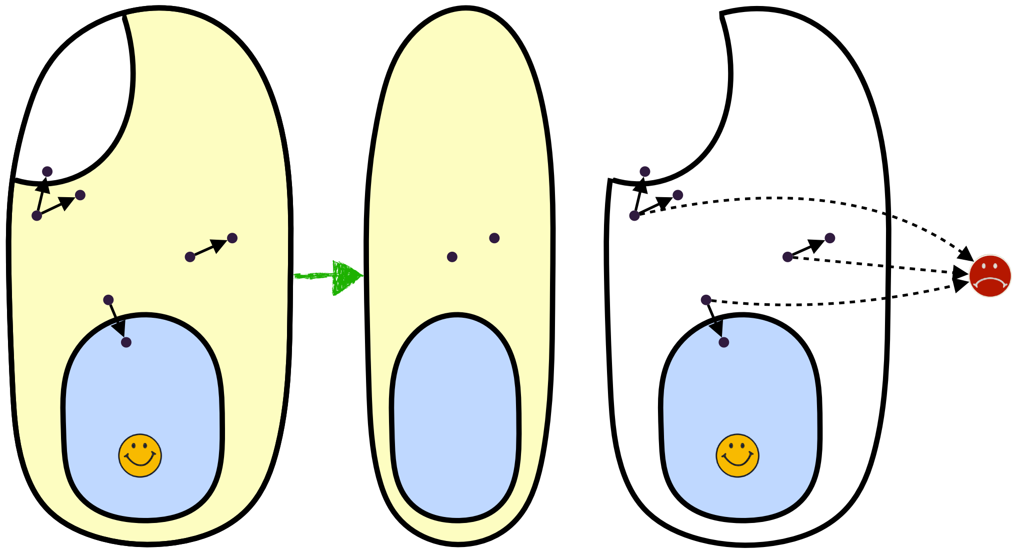


Further properties of the biased Markov chain obtained via an abstraction

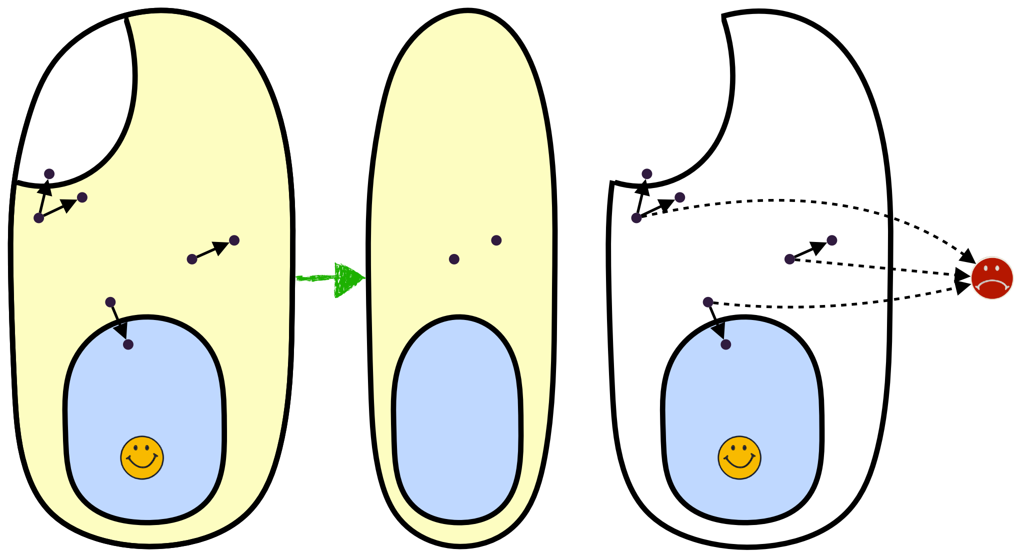


Further properties of the biased Markov chain obtained via an abstraction

Property of the bias

$$\gamma(\rho) = \begin{cases} \mu^\bullet(\alpha(s_0)) & \text{if } \rho \text{ ends in } ☺ \\ 0 & \text{otherwise} \end{cases}$$

It is bivaluated, hence bounded



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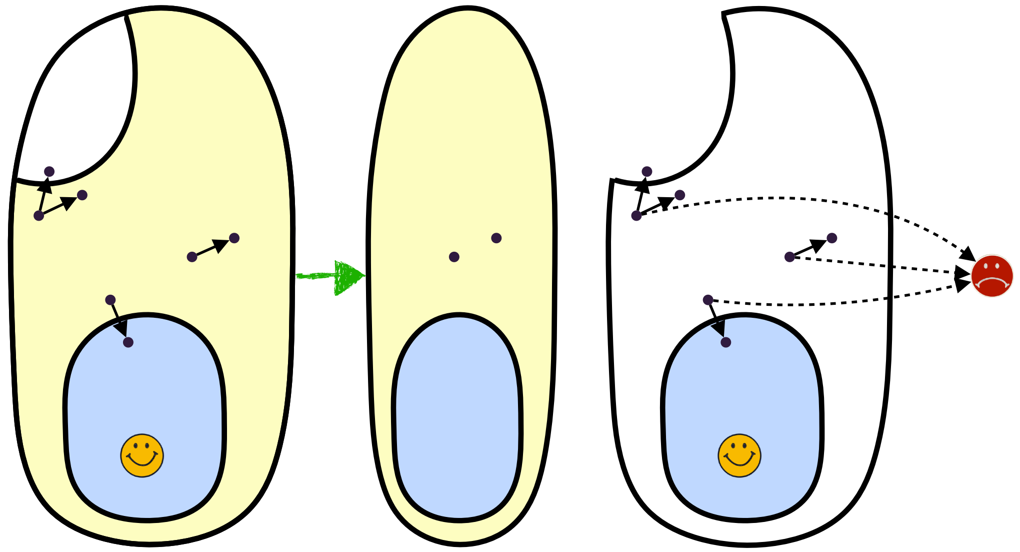

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If F is finite and for every $0 \leq x < 1$, $\{s \in S \mid \mu^\bullet(\alpha(s)) \geq x\}$ is finite, then $\mathscr{C}'$ is decisive w.r.t. ☺.

Proof using attractors, martingale theory

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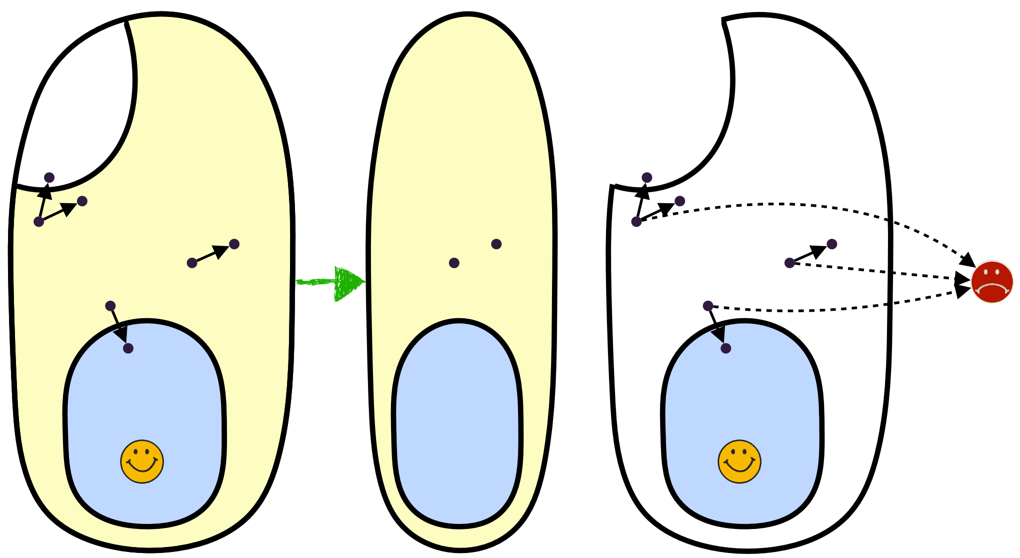

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- The analysis can be performed on $\mathscr{C}'$!

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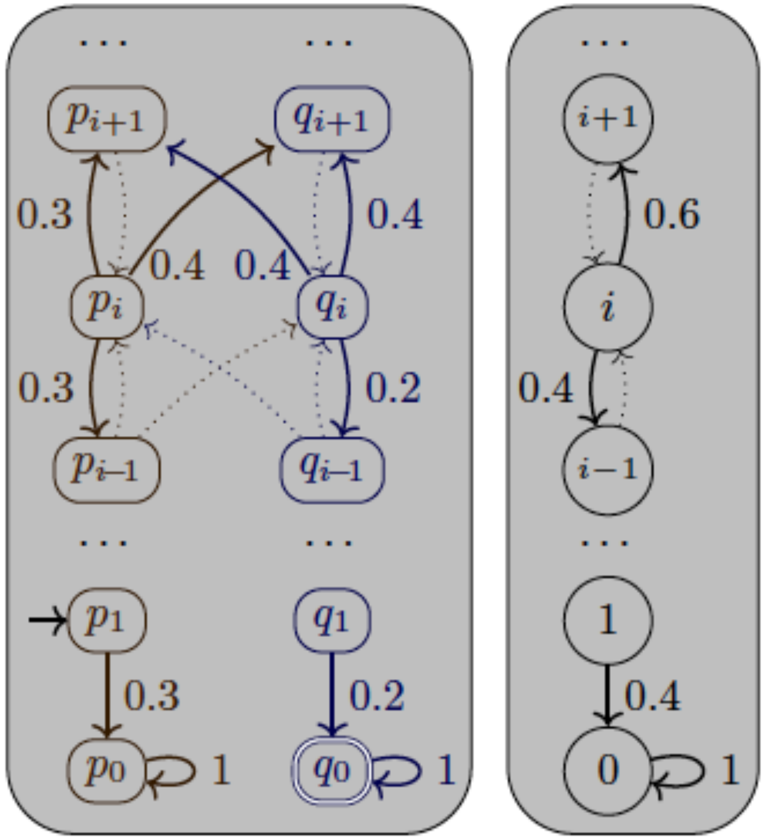

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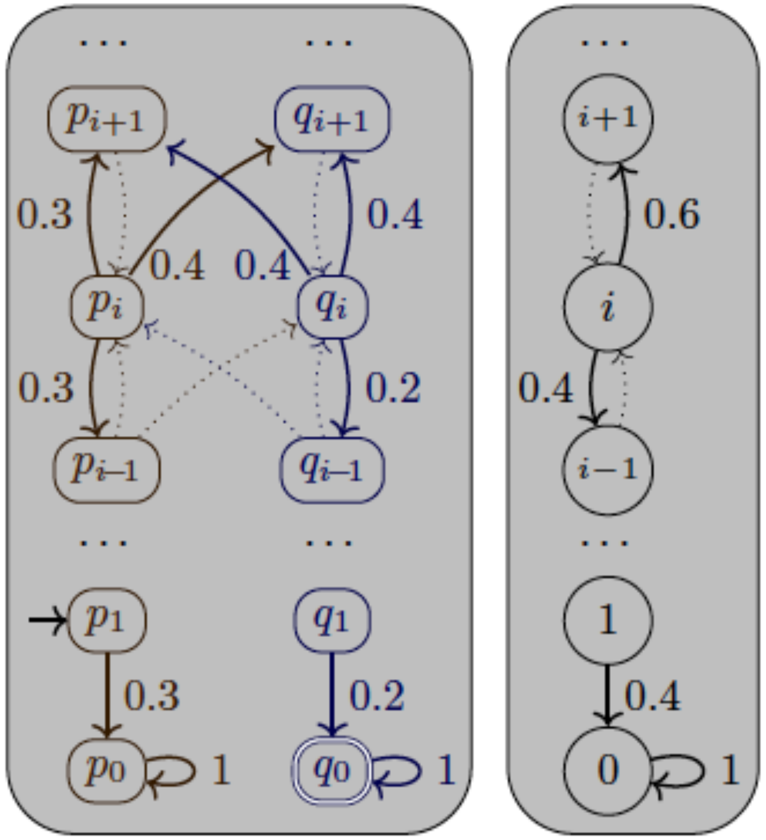
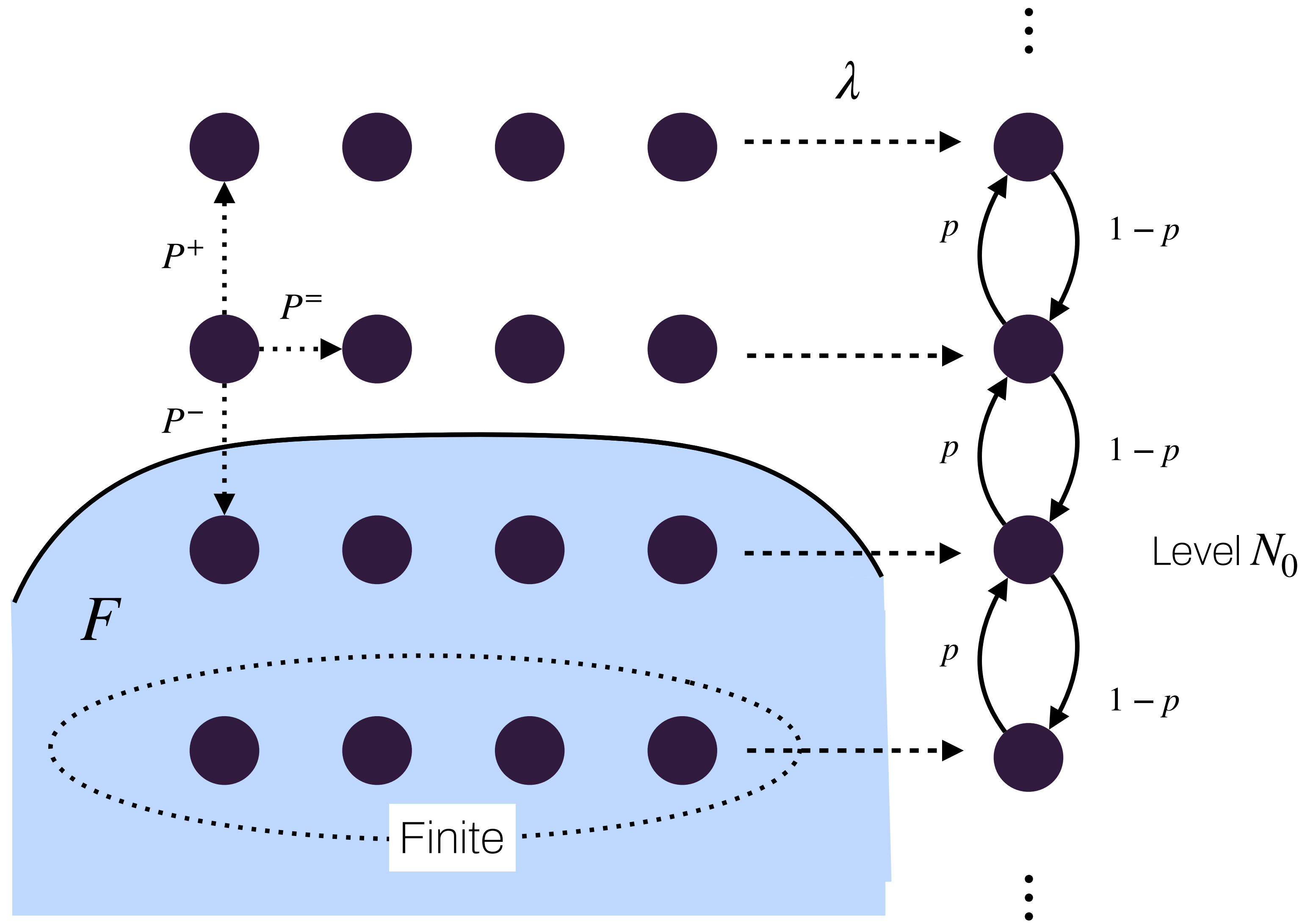


Only one condition needs to be satisfied...
The monotony condition!

Condition for monotony

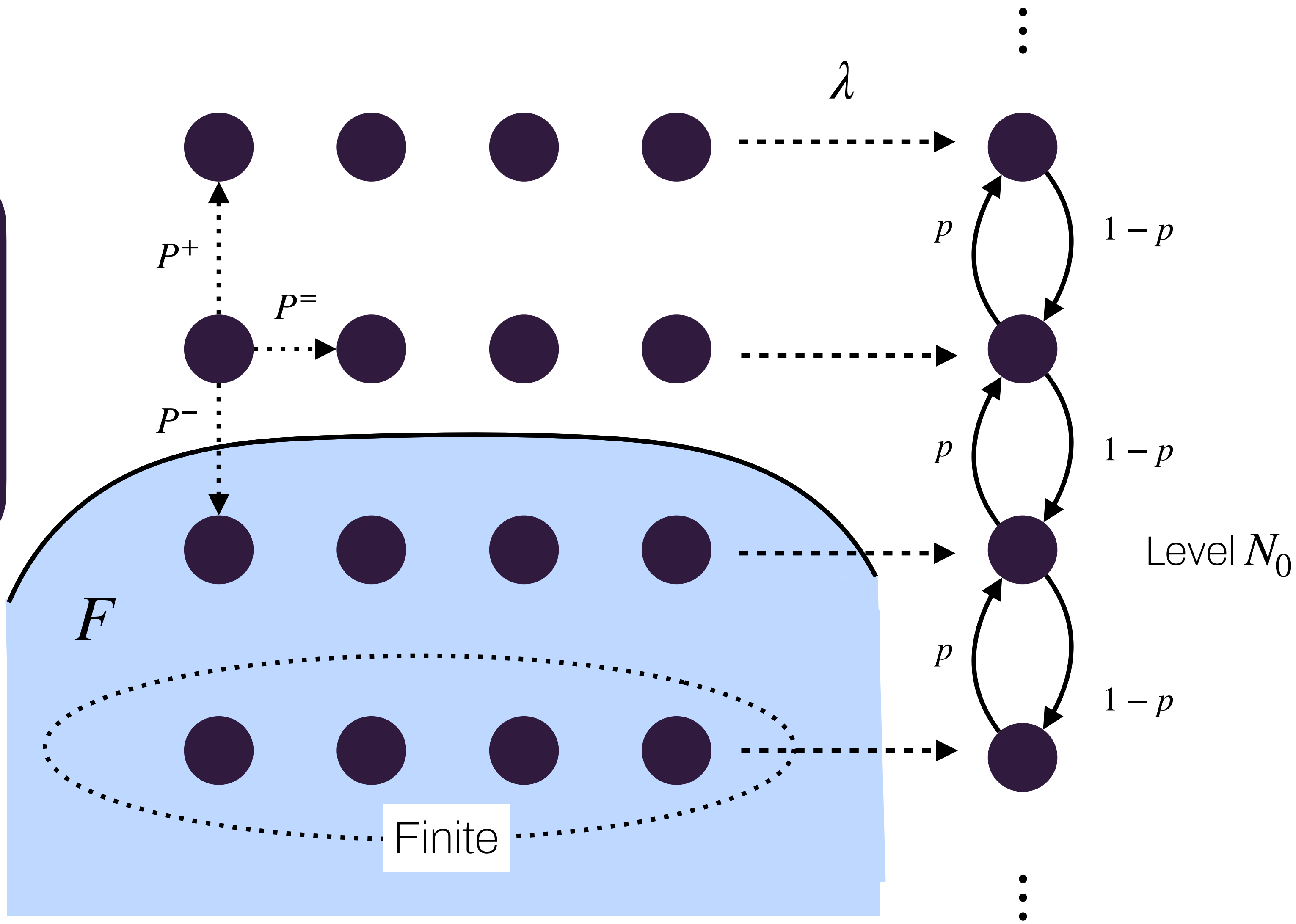


Condition for monotony

$(\hat{p}, N_0)$-divergence

For every $s \notin F$, $P^{=}(s) < 1$ implies

$$\frac{P^{+}(s)}{P^{-}(s) + P^{+}(s)} \geq \hat{p}$$



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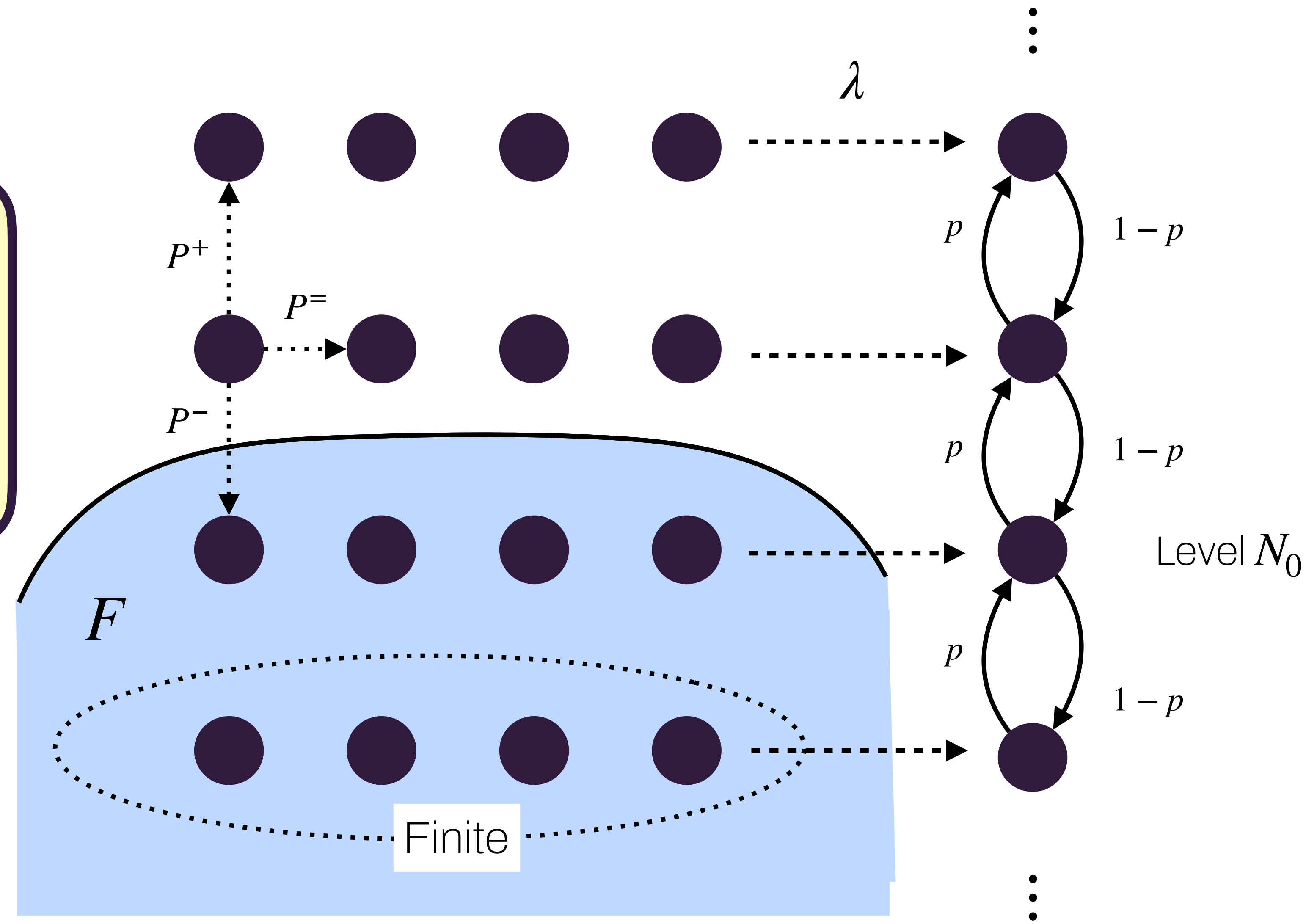

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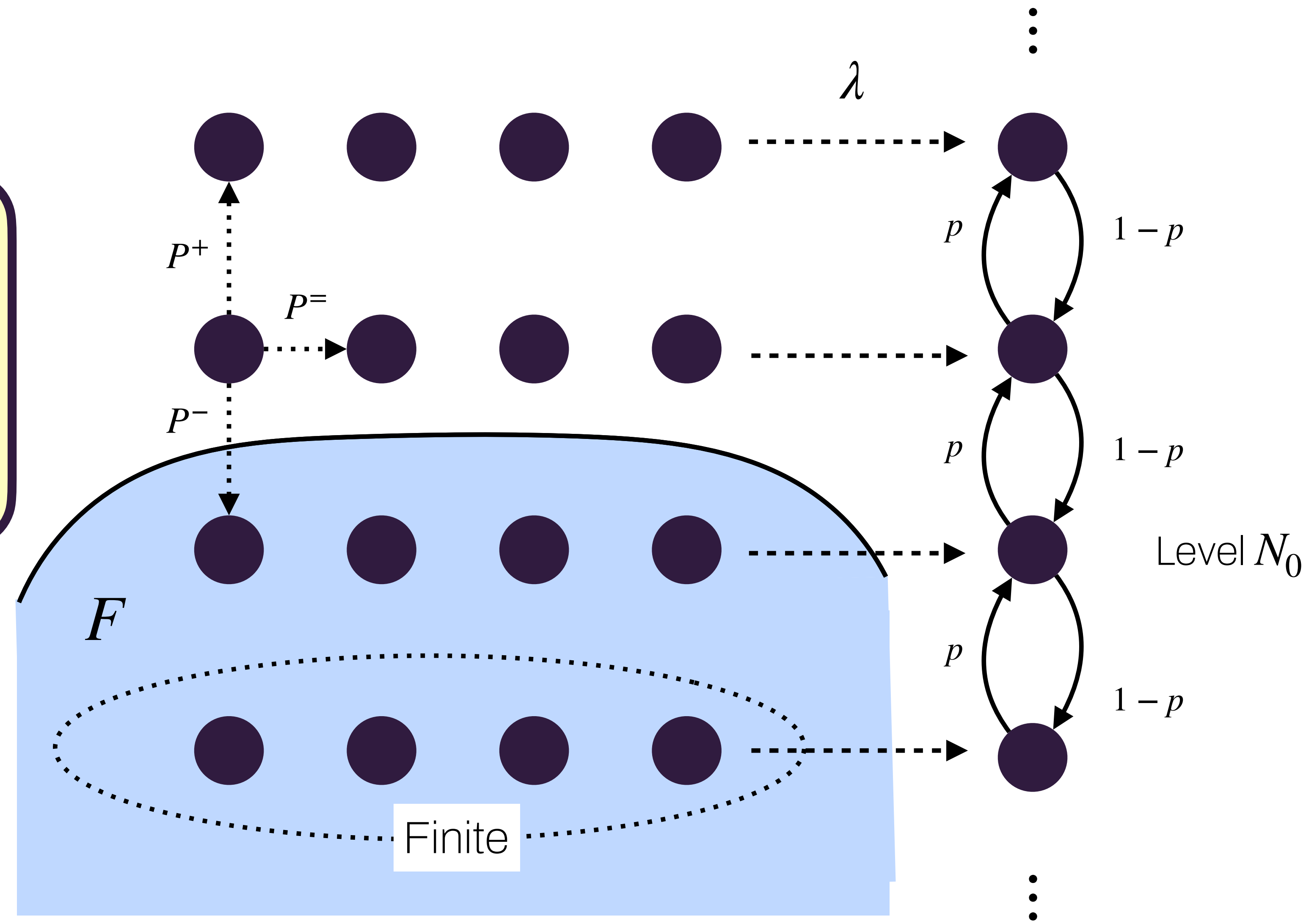


$(\hat{p}, N_0)$-divergence of $\mathscr{C}$ with $\frac{1}{2} < p < \hat{p}$ implies the random walk $\mathscr{C}_p^{\bullet}$ is an abstraction of $\mathscr{C}$

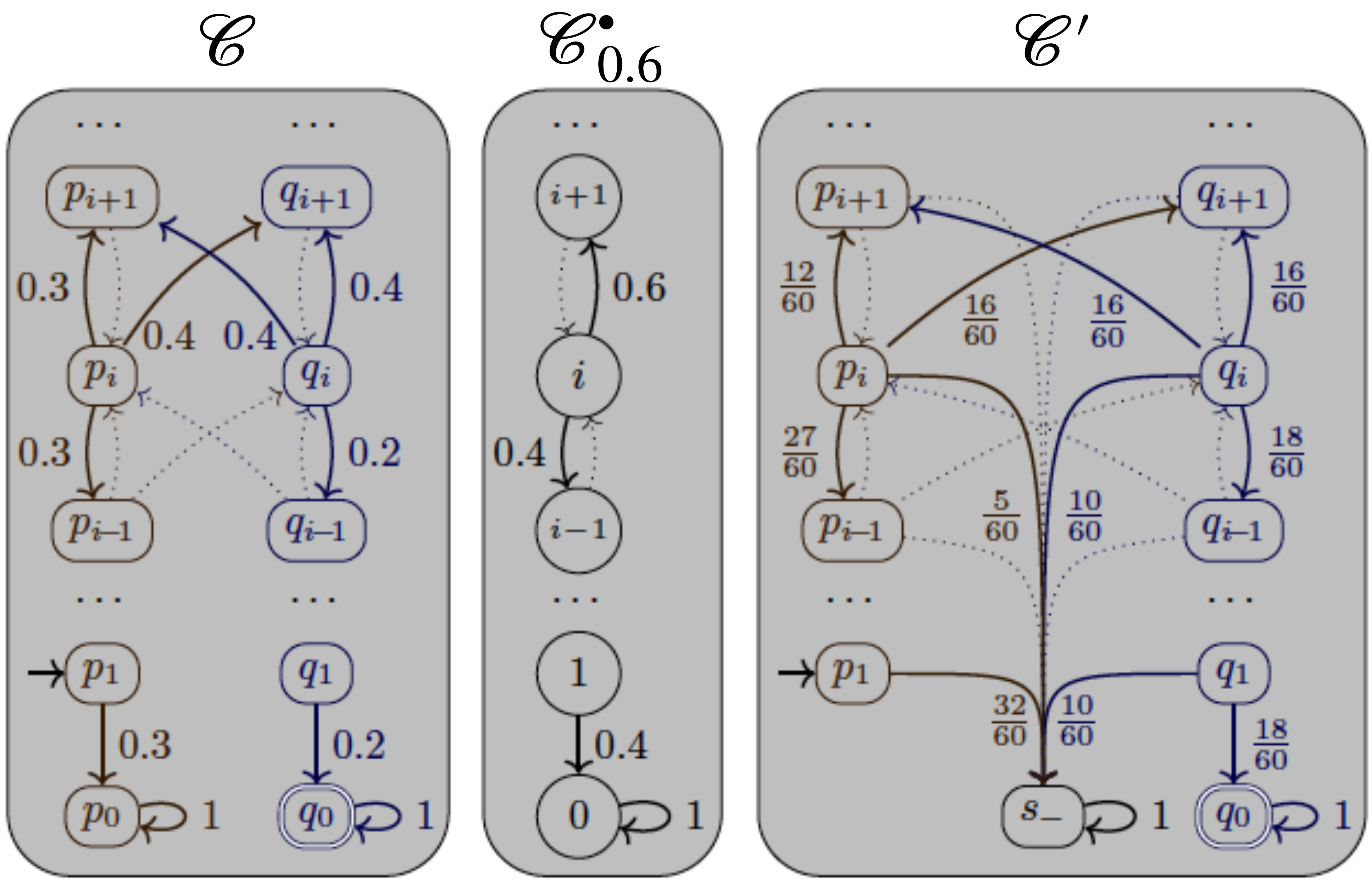
Correctness of the approach

Theorem

Let $\mathscr{C}$ be an LMC with level function λ, $\mathscr{C}_p^\bullet$ the random walk of parameter p. Assume there is N_0 and $\hat{p}$ s.t. $\frac{1}{2} < p < \hat{p}$ and $\mathscr{C}$ is $(\hat{p}, N_0)$-divergent.Then:

- The analysis of $\mathscr{C}$ can be made via the biased Markov chain obtained by importance sampling through the abstraction $\mathscr{C}_p^\bullet$
- If furthermore, $\inf_{s \text{ s.t. } \lambda(s)>N_0} P^+(s) > 0$, then the expected time to sample an execution is finite

Example



$\mathscr{C}$ is not decisive

$\mathscr{C}'$ is decisive
+ finite sampling time

Implementation and experiments

https://cosmos.lacl.fr/
[BBDHP15] P. Ballarini, B. Barbot, M. Duflot, S. Haddad, N. Pekergin. Hasl: A new approach for performance evaluation and model checking from concepts to experimentation (Performance Evaluation)

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Note: in all experiments, the confidence is set to $99\ \%$

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First example

- State-free proba. pushdown automaton $\mathscr{C}$: $\{A \xrightarrow{1} C; A \xrightarrow{n} BB; B \xrightarrow{5} \varepsilon; B \xrightarrow{n} AA; C \xrightarrow{1} C\}$
- Start from A, and target the empty stack

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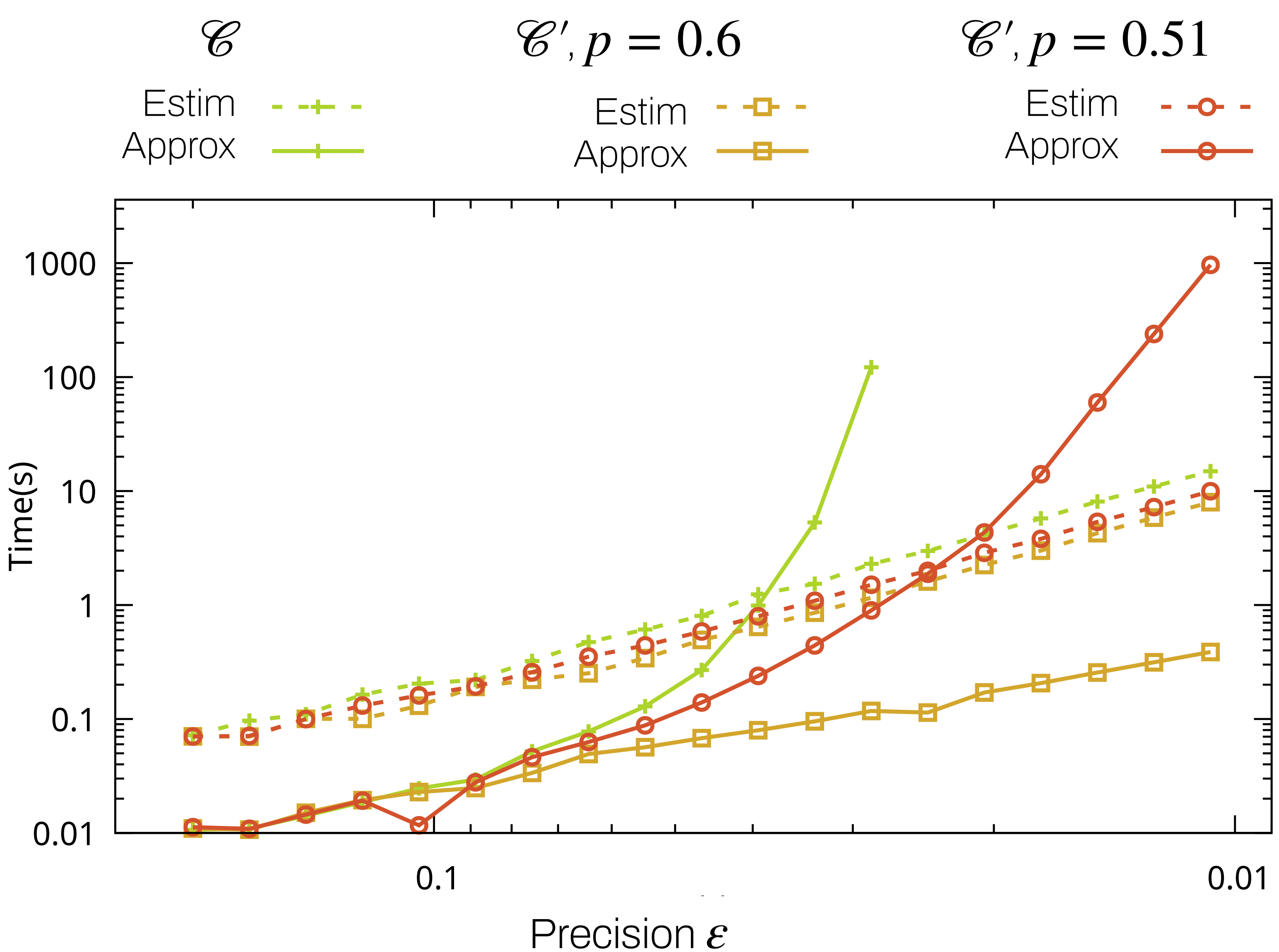

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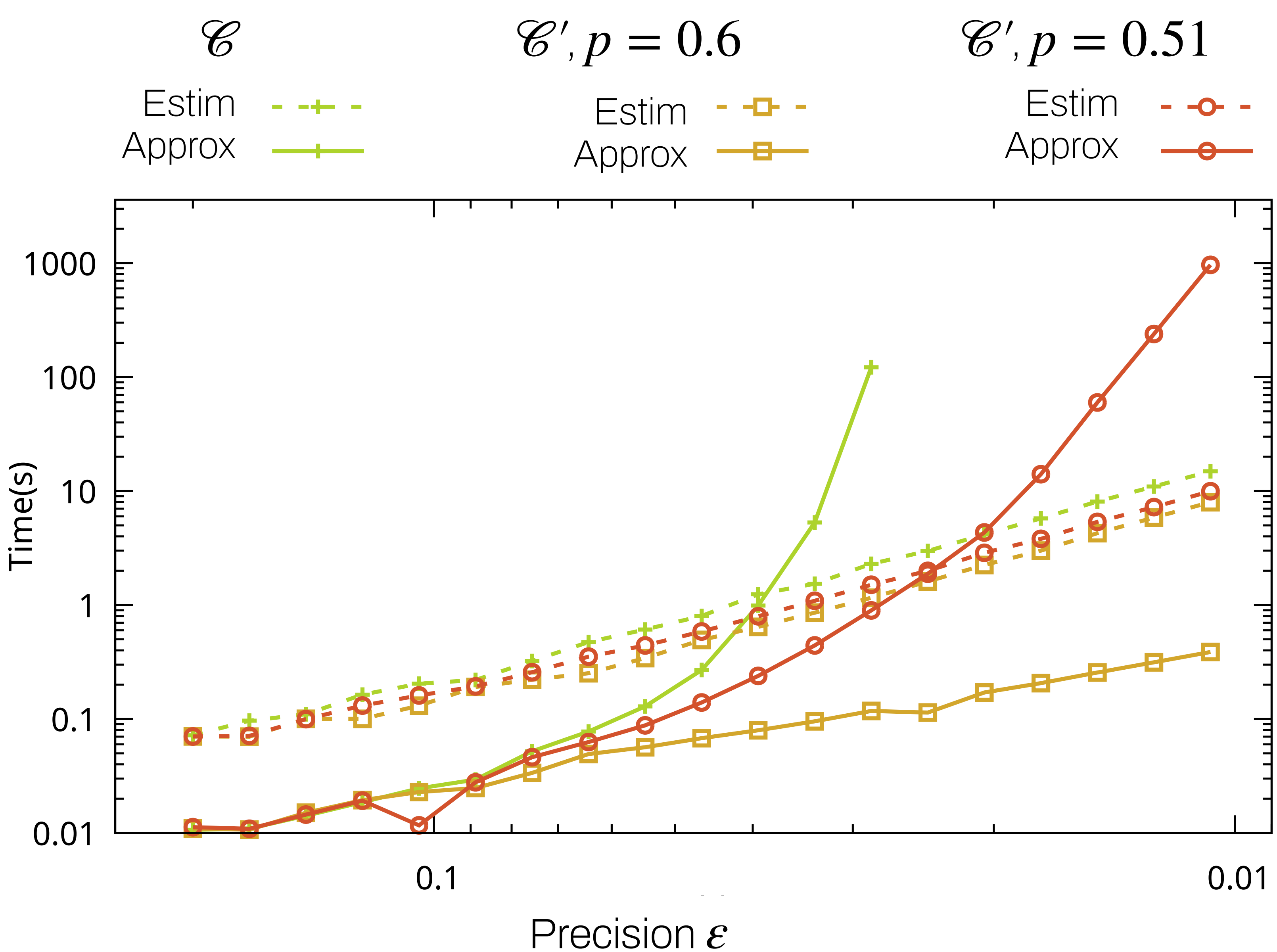
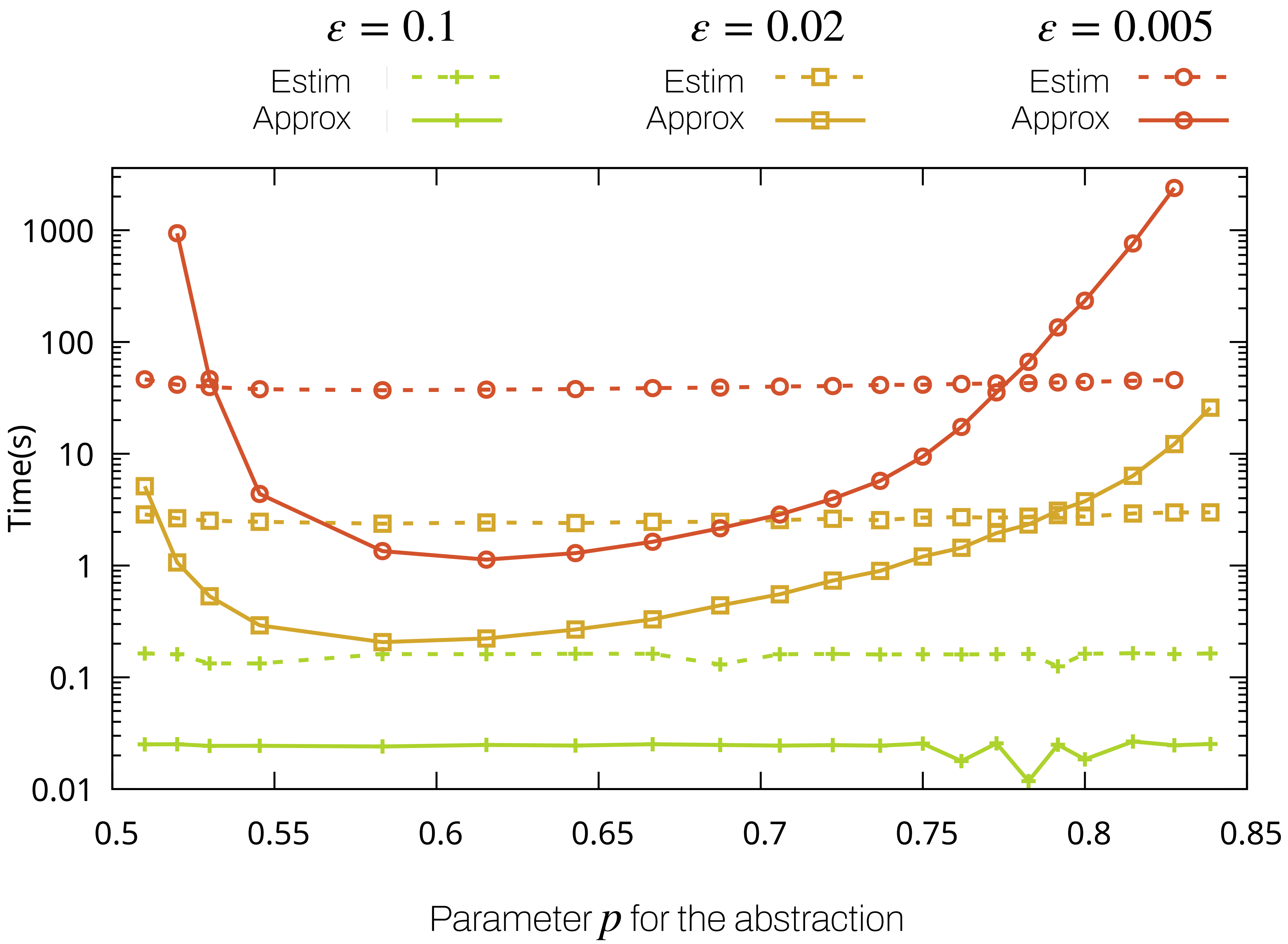


- In Estim (SMC): doubling the precision impacts in square on computation time (slope 2 in this log-log scale)
- Importance sampling seems to improve the analysis time, both for Approx and Estim (no formal guarantee, though)
- There seems to be « a best p » ($p = 0.6$ here)
- For that best p, Approx behaves very well!

First example — continued



Second example

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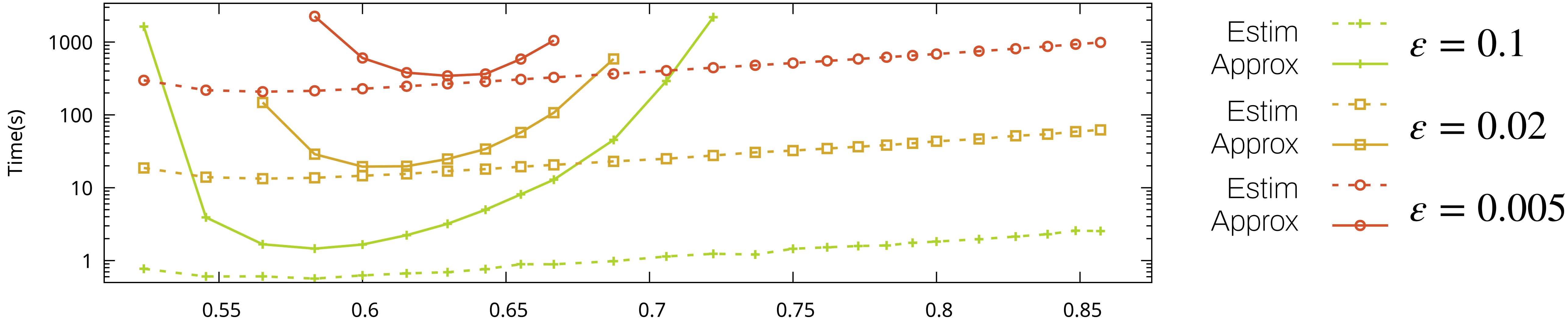

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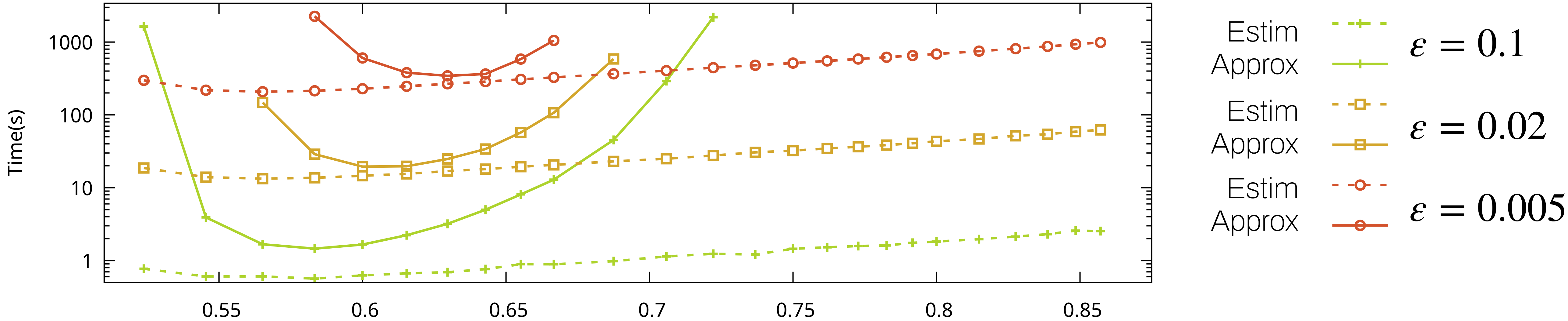

Parameter p for the abstraction

- Estim (SMC) not too sensitive to p
  - Neverthess (log scale): clear bell effect on p
- Approx very sensitive to p

Conclusion

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- p big: large desactivation zone (N_0)
- p small: small bias (few trajectories end up in ☹)