Tightening the Frontier of Decidability for Decisiveness

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Plan

Markov Chains

A Markov Chain (MC) $\mathcal{M} = (S, p)$ is defined by:

- S, a countable set of states;

$$\textcircled{0} \quad \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \cdots$$

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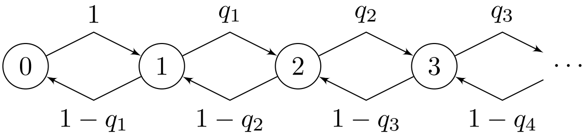

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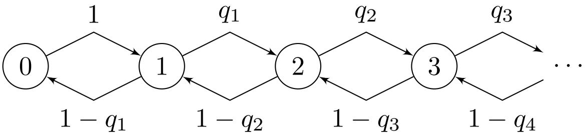

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- for all $s \in S$, the support of $p(s)$ is finite and computable;
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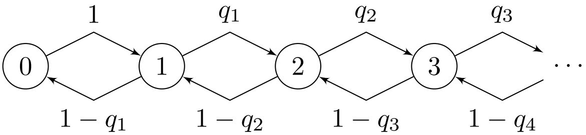

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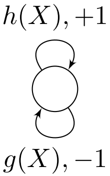
Here $i \to q_i$ should be computable.

High-Level Models for Markov Chains



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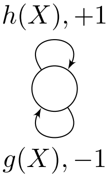

1-state and 1-counter machine

$t_{incre} : q \xrightarrow{h(X),+1} q$: X':=X+1

$t_{decre} : q \xrightarrow{g(X),-1} q$: if X>0, then X':=X-1

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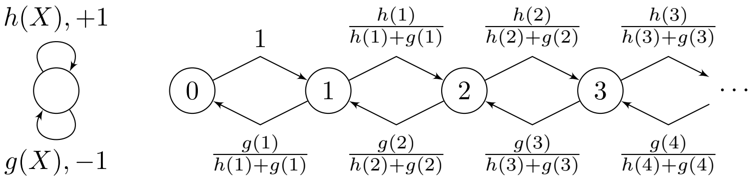

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- given a state, getting the probabilities
  by normalization of the weights of the enabled transitions.

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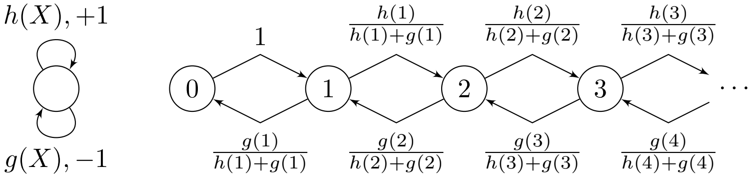

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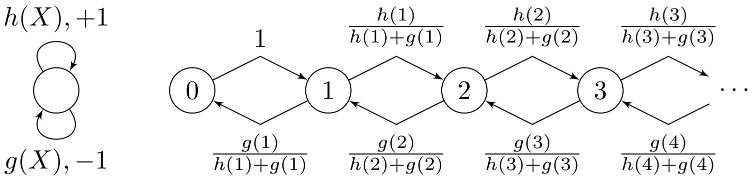

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The weights are static if h and g are constant.

Computing Reachability Probabilities

Let $\mathcal{M}$ be a Markov chain, s_0 an (initial) state, and A a subset of states, then $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F}A)$ represents the probability to reach A from s_0.

The Computing Reachability Probability (CRP) problem is defined by:

- Input: effective $\mathcal{M}$, s_0, effective A, and a rational number $\theta > 0$;
- Output: an interval $[low, up]$ such that $up - low \leq \theta$ and $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F}A) \in [low, up]$.

In finite Markov chains, it can be solved in polynomial time.

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How to solve the CRP problem of infinite Markov chains?

- ad-hoc algorithms for particular class of probabilistic models, *e.g.*, static *Probabilistic Pushdown Automata (pPDA)* (Brádzil et al, FMSD 2013);
- generic algorithms for probabilistic models satisfying a semantical property, *e.g.*, *decisiveness* (Abdulla et al, LMCS 2007).

Motivation of previous and current work

(Finkel et al. CONCUR 2023 and this work)

Limitations of existing approaches

- models with only constants (static) transition weights cannot model phenomena like congestion in networks (Abdulla et al, LMCS 2007);

Our contributions

- models may contain *dynamic* weights;

Motivation of previous and current work

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Limitations of existing approaches

- models with only constants (static) transition weights cannot model phenomena like congestion in networks (Abdulla et al, LMCS 2007);
- the decisiveness problem for some standard models are not yet studied.

Our contributions

- models may contain *dynamic* weights;
- Decidability results of decisiveness problem for dynamic probabilistic counter machines and Petri nets.

Plan

Decisiveness

$\mathcal{M}$ is *decisive* w.r.t. $s_0 \in S$ and $A \subseteq S$ if almost surely a run starting from s_0:

- either reaches A;
- or $\overline{\mathbf{EF}A}$, i.e., the set of states from which A is unreachable.

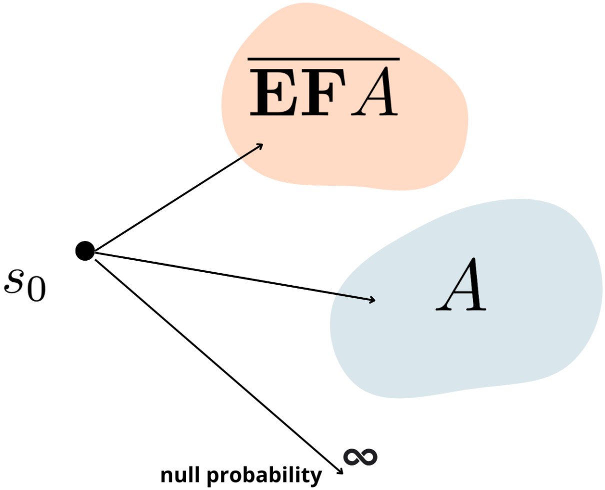

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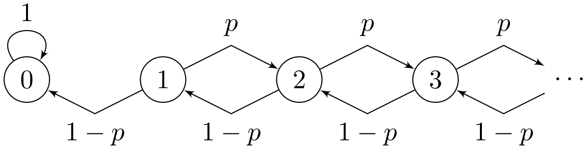
Illustration with a polynomial random walk: $s_0 = 1$ and $A = \{0\}$



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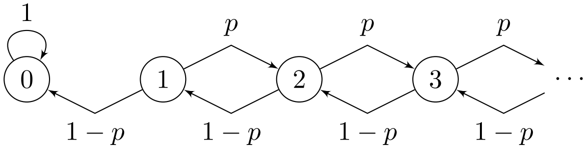

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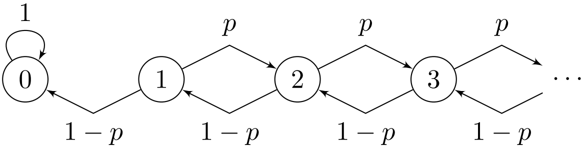

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The decisiveness problem w.r.t. s_0 and finite A for polynomial safe one-counter pCM with a single state is decidable in linear time. (Finkel et al., CONCUR 2023)

Research on decisiveness

Decisiveness is studied via different approaches:

- In relationship to statistical model checking (Barbot et al., 2024)
- Extension of the notion of decisiveness for MDPs (Bertrand et al., 2020)
- Study of decisiveness for stochastic hybrid systems (Bouyer et al., 2022)

Plan

From QBD to pCM

Quasi Birth-Death Process (QBD) is a probabilistic model
widely used and analyzed in performance evaluation.

It is equivalent to a probabilistic 1-counter machine with following constraints:

- Counter updates are incrementations and decrementations;
- For all states q, q', positive integers n, n' and $\Delta \in \{-1, 0, 1\}$

$$\mathbf{Pr}((q, n), (q', n + \Delta)) = \mathbf{Pr}((q, n'), (q', n' + \Delta))$$

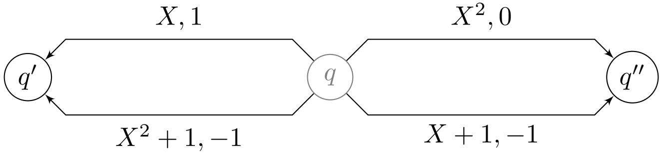
.

From pCM to pHM

A probabilistic Homogenous 1-counter Machine (pHM) is an extension of QBD:

- the weights are polynomials whose single variable X is the counter value;

Illustration. *(The transitions for the null counter are omitted)*



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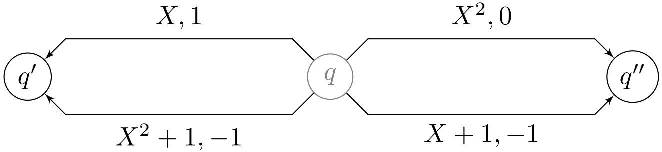

Consider the Markov chain over $Q \times \mathbb{N}$, denoted as $\mathbf{M}_{Q,\mathbb{N}}$, then

$$\mathbf{M}_{Q,\mathbb{N}}[(q, \nu), ((q', \nu + 1))] = \frac{\nu}{2(\nu^2 + \nu + 1)}$$

From pCM to pHM

A probabilistic Homogenous 1-counter Machine (pHM) is an extension of QBD:

- the weights are polynomials whose single variable X is the counter value;
- but the coefficients of $\mathbf{M}_Q$ are still constant:

$$\mathbf{M}_Q(q, q') = \frac{\sum_{t=(q \to q') \in \Delta_>} W(t)}{\sum_{t=(q \to) \in \Delta_>} W(t)}$$

where $\Delta_>$ are the transitions for the positive counter

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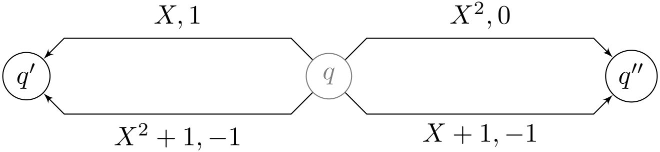

$$\mathbf{M}_Q[q, q'] = M_Q[q, q''] = \frac{X^2 + X + 1}{2(X^2 + X + 1)} = \frac{1}{2}$$

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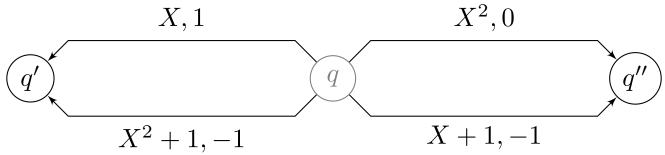
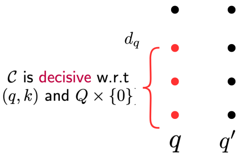


The decisiveness problem of a pHM with irreducible $\mathbf{M}_Q$ is decidable in polynomial time (Finkel et al. CONCUR 2023).

The decisiveness problem of a pHM is decidable in polynomial time (here).

How to solve the CRP for pHM?

- $d_q = \sup\{k \mid \mathcal{C}$ is decisive w.r.t. (q,k) and target $Q \times \{0\}\}$
- $d_q \in \mathbb{N} \cup \{\infty\}$



> Let $\mathcal{C}$ be a pHM. Then the decisiveness problem w.r.t $(q_0, k_0) \in Q \times \mathbb{N}$ and $Q \times \{0\}$ is decidable in polynomial time.

Computing d_q solves the CRP problem.
(but not necessarily the reverse)

Can one compute d_q and in the positive case with what complexity?

Proof sketch (1)

- $r_q = \sup\{k \mid Q \times \{0\}$ is reachable from$(q, k)\}$
- $Q_f = \{q \in Q \mid r_q < \infty\}$ and $Q_\infty = Q \setminus Q_f$.

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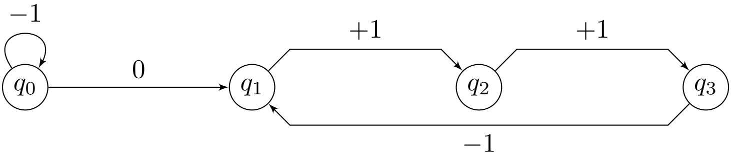

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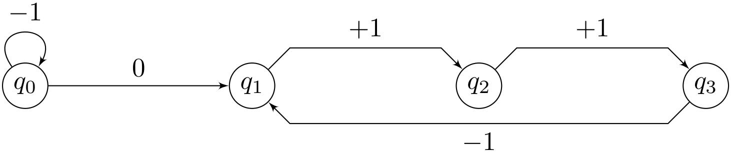

- $r_{q_0} = \infty$; $r_{q_1} = 0$; $r_{q_2} = 0$; $r_{q_3} = 1$
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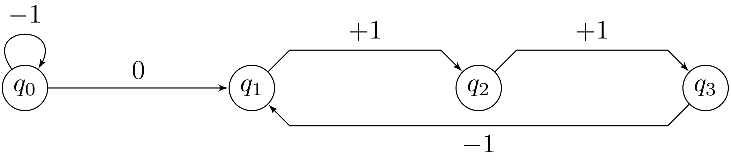

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Let $\mathcal{C}$ be a pHM. One can compute in polynomial time $(r_q)_{q \in Q}$.

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- Every BSCC is viewed as an irreducible Markov chain.
  Thus for such q, d_q is computed in polynomial time (Finkel et al., CONCUR 2023).

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Let Y_f (resp. Y_∞) be the set of states q of the BSCCs of $\mathbf{M}_Q$
such that $d_q < \infty$ (resp. $d_q = \infty$).

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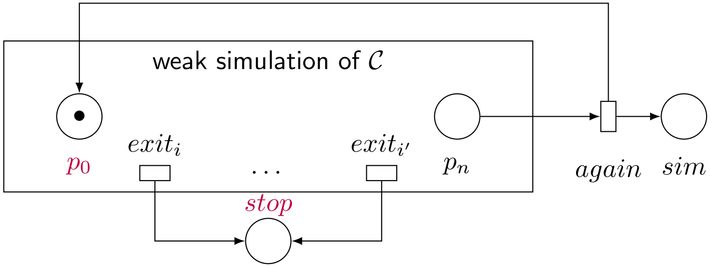
- From every SCC Z such that Y_f is unreachable, almost surely one reaches Y_∞.
  Thus for all $q \in Z$, $d_q = \infty$.
- From every SCC Z such that Y_f is reachable then for all $q \in Z$, $d_q < \infty$ and we have designed a polynomial time algorithm to compute such d_q.

Plan

Polynomial Probabilistic Petri Nets: Decisiveness

The decisiveness problem of polynomial pPNs
w.r.t. an upward closed set is undecidable (Finkel et al. CONCUR 2023).

Sketch of Proof.



By reduction of the halting problem for a *normalized* counter machine $\mathcal{C}$.

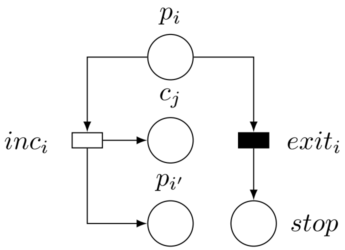
A normalized CM resets the counters at the start and the end of the computation.

The probabilistic Petri net infinitely repeats a weak simulation for $\mathcal{C}$ incrementing a counter of simulations sim (single variable of the polynomial weights), with at each instruction some (variable) probability to exit the simulation.

Sketch of Proof (continued)

Simulation of an incrementation

$$i : c_j \leftarrow c_j + 1; \textbf{goto } i'$$



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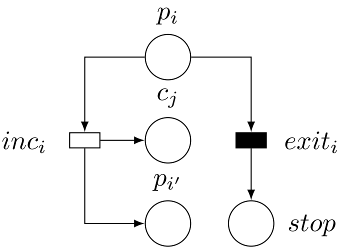
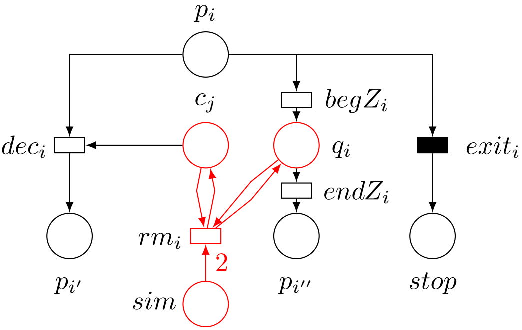


Simulation of a decrementation

$$\begin{array}{l} i : \textbf{if } c_j > 0 \textbf{ then} \\ \quad c_j \leftarrow c_j - 1; \textbf{goto } i' \\ \textbf{else} \\ \quad \textbf{goto } i'' \end{array}$$



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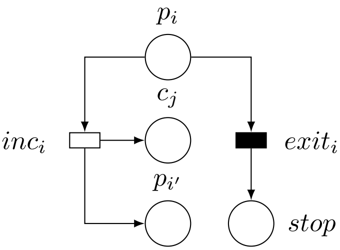

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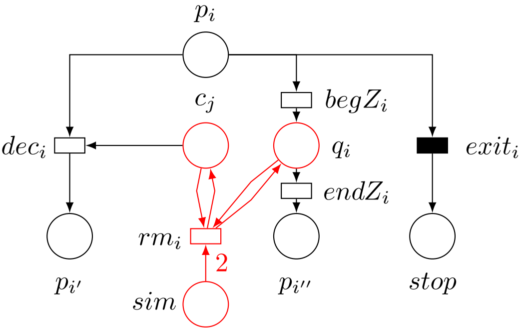

When cheating the net is punished by a possible decrementation of sim.

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Due to the choice of the polynomial weights, when sim goes to infinity,

- If i is an incrementation, $W(exit_i) = o(W(inc_i))$;
- If i is a decrementation, $W(exit_i) = o(W(begZ_i))$ and $W(begZ_i) = o(W(dec_i))$.

Thus the more the simulations are achieved without cheating

the less probable the net will stop or cheat.

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Assume that $\mathcal{C}$ halts.

The infinite path corresponding to the repetition of the correct simulation of $\mathcal{C}$

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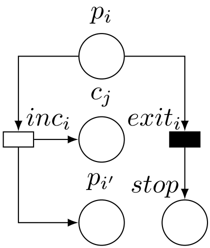
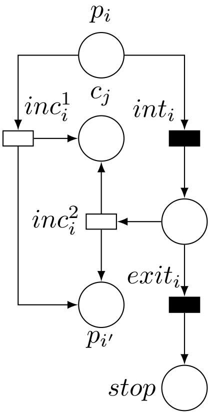
The analysis of the case when $\mathcal{C}$ does not halt is much more involved.

From Polynomial Weights to Affine Weights

Incrementation





$$W(inc_i)(\mathbf{m}) = (\mathbf{m}(sim) + 2)^2 - 1 \qquad W(inc_i^j)(\mathbf{m}) = (\mathbf{m}(sim) + 1)$$

$$\mathbf{Pr}(p_i, stop) = 1/(\mathbf{m}(sim) + 2)^2 \qquad \mathbf{Pr}(p_i, stop) = 1/(\mathbf{m}(sim) + 2)^2$$

Plan

Conclusion and Perspectives

Contributions

- Establish the decidability result of decisiveness for pHM without the irreducibility hypothesis;
- Establish the undecidability result of decisiveness for probabilistic Petri nets with affine weights.

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Perspectives

- Study the decidability of decisiveness of static pPNs w.r.t. arbitrary finite set;
- Establish sufficient conditions for decisiveness for models with undecidability of decisiveness;
- Examine the relationship between decisiveness and divergence introduced in (Finkel et al, RP 2023).